Thermodynamics of binary black holes and neutron stars

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We consider compact binary systems, modeled in general relativity as vacuum or perfect-fluid spacetimes with a helical Killing vector k^α, heuristically, the generator of time-translations in a corotating frame. Systems that are stationary in this sense are not asymptotically flat, but have asymptotic behavior corresponding to equal amounts of ingoing and outgoing radiation. For black-hole binaries, a rigidity theorem implies that the Killing vector lies along the horizon's generators, and from this one can deduce the zeroth law (constant surface gravity of the horizon). Remarkably, although the mass and angular momentum of such a system are not defined, there is an exact first law, relating the change in the asymptotic Noether charge to the changes in the vorticity, baryon mass, and entropy of the fluid, and in the area of black holes.

Binary systems with $M\Omega$ small have an approximate asymptopia in which one can write the first law in terms of the asymptotic mass and angular momentum. Asymptotic flatness is precise in two classes of solutions used to model binary systems: spacetimes satisfying the post-Newtonian equations, and solutions to a modified set of field equations that have a spatially conformally flat metric. (The spatial conformal flatness formalism with helical symmetry, however, is consistent with maximal slicing only if replaces the extrinsic curvature in the field equations by an artificially tracefree expression in terms of the shift vector.) For these spacetimes, nearby equilibria whose stars have the same vorticity obey the relation $\delta M = \Omega\delta J$, from which one can obtain a turning point criterion that governs the stability of orbits.

I. INTRODUCTION

Beginning with papers by Blackburn and Detweiler [1,2], several authors have used spacetimes with a helical Killing vector [1] to model binary systems in the context of general relativity. Such spacetimes can be regarded as having equal amounts of incoming and outgoing radiation; they are a counterpart in general relativity of the stationary solution due to Schild [4] that describes two oppositely charged particles whose electromagnetic field is the half-advanced + half-retarded solution of the orbiting charges. Because the radiation field of such a stationary solution has infinite energy, spacetimes that describe the corresponding general relativistic binaries are not asymptotically flat. Instead, the asymptotic mass rises linearly with a naturally defined radial coordinate.

The formal lack of asymptotic flatness has been handled in several related ways. As Detweiler has emphasized, one can define an approximate asymptotic region for systems in which the energy emitted in gravitational waves in a dynamical time is small compared to the mass of the system. In this "local wave zone," the geometry describes gravitational waves propagating on a Schwarzschild background. In the more restrictive context of the post-Newtonian approximation, one regains asymptotic flatness, because there is no radiation through second post-Newtonian order. Finally, a number of authors [5–9] considered spacetimes with conformally flat spacelike slices that satisfy a truncated set of field equations consisting of the initial value equations, a field equation for the spatial conformal factor, and the equation of hydrostatic equilibrium. Like the post-Newtonian spacetimes, these conformally flat spacetimes are nonradiative and (as we show) are asymptotically flat.

We consider binary systems modeled in the exact theory (without asymptotic flatness) and then apply our results to the post-Newtonian spacetimes and spatially conformally flat spacetimes that retain asymptotic flatness. In each case one uses the Killing vector, k^α, to define a conserved current and associated charges. For the exact vacuum and perfect-fluid spacetimes, the Noether current of the helical Killing vector assigns to each spacetime a charge Q. (See,

[1] In a spacetime with a rotational Killing vector ϕ^α and a timelike Killing vector t^α, each combination $t^\alpha + \Omega\phi^\alpha$, with Ω constant and nonzero, will be called a helical (or helicoidal) Killing vector. We give a precise definition in II and discuss its relation to a previous definition by Bonazzola et al [3]

for example, Refs. [10–15].) Despite the lack of asymptotic flatness one can choose the current to make Q finite, and it Q is independent of the 2-surface S on which it is evaluated, as long as S lies outside the matter and all black holes. The Noether current assigns to each black hole a charge that can be identified with its entropy (its area, in the spacetimes we consider); and we obtain a version of the first law (Eq. (55) below) that expresses the change δQ in terms of changes in the vorticity, baryon mass, and entropy of the fluid, and in the area of black holes. Independent work by Baker and Detweiler [16] obtains a similar first law for spacetimes with approximate asymptotic flatness at finite distance from the binary.

In the asymptotically flat spacetimes mentioned above, the helical Killing vector has the asymptotic form $k^\alpha = t^\alpha + \Omega\phi^\alpha$, where t^α and ϕ^α generate asymptotic symmetries associated with time-translations and rotations. Neighboring perfect-fluid equilibria in a post-Newtonian or a spatially conformally flat framework satisfy a first law of thermodynamics of the form

$$\delta M = \Omega\delta J + \int_\Sigma \left[\bar{T}\Delta dS + \bar{\mu}\Delta dM_{\rm B} + v^\alpha \Delta dC_\alpha\right] + \sum_i \frac{1}{8\pi}\kappa_i \delta A_i.$$

Here M and J are the ADM mass and angular momentum of the spacetime (see Eqs. (108,109); $\bar{T}$ and $\bar{\mu}$ are the redshifted temperature and chemical potential; dM_B is the baryon mass of a fluid element; and dC_α is related to the circulation of a fluid element (see Eqs. (56,57)).

Note that, in the full theory, models of binaries with a helical Killing vector can only have corotating black holes. If their generators do not lie along the Killing vector the black holes will have nonzero shear and thus (assuming positive energy) increasing area; and this is inconsistent with the assumption of a helical Killing vector. In an appendix, we derive a virial relation for binary neutron-star systems in a conformally flat framework and show that the relation is equivalent to the equality of the Komar and ADM mass.

One other class of asymptotically flat spacetimes with a single Killing vector is worth mentioning. These are non-axisymmetric stars whose figure is *stationary in an inertial frame*, the analog in general relativity of the Newtonian Dedekind ellipsoids. We expect that such stationary, nonaxisymmetric perfect-fluid spacetimes exist; their velocity fields have nonzero shear, however, and cannot be stationary when viscosity is present. [17]

Conventions: Spacetime indices are Greek, spatial indices Latin, and the metric signature is $-+++$. Readers familiar with abstract indices can regard indices early in the alphabet as abstract, while i, j, k, l are concrete, associated with a chart $\{x^i\}$. We use the dual form of Stokes' theorem for the divergence of an antisymmetric tensor $A^{\alpha\cdots\beta\gamma}$, namely

$$\int_\Sigma \nabla_\gamma A^{\alpha\cdots\beta\gamma} dS_{\alpha\cdots\beta} = \int_{\partial\Sigma} A^{\alpha\cdots\beta\gamma} dS_{\alpha\cdots\beta\gamma}$$

where $dS_\alpha = \epsilon_{\alpha\beta\gamma\delta} dS^{\beta\gamma\delta}$, $dS_{\alpha\beta} = \epsilon_{\alpha\beta\gamma\delta} dS^{\gamma\delta}$. For example, in an oriented chart $t, \{x^i\}$ with Σ a surface of constant t and $\partial\Sigma$ a surface of constant t and x^1, $dS_\alpha = \nabla_\alpha t\sqrt{-g}d^3x$, $dS_{\alpha\beta} = \frac{1}{2}(\nabla_\alpha t \nabla_\beta x^1 - \nabla_\alpha x^1 \nabla_\beta t)\sqrt{-g}d^2x$. Finally, if S is a 2-surface in a 3-space Σ and ϵ_{abc} is the volume form on Σ associated with a 3-metric γ_{ab}, we write $dS_a = \epsilon_{abc} dS^{bc}$; for S a surface of constant r, $dS_a = \nabla_a r\sqrt{\gamma}d^2x$.

II. HELICAL KILLING VECTORS, EVENT HORIZONS, AND THE ZEROTH LAW

We consider globally hyperbolic spacetimes $M, g_{\alpha\beta}$ that have a symmetry vector k^α, a Killing vector that generates a symmetry of the matter fields. Our particular interest is in stationary binary systems, systems whose Killing vector k^α has helical integral curves with a fixed period T; but our results hold for a broader class of spacetimes with a single Killing vector.

We begin by using the periodic orbits just mentioned to define a *helical* vector field. We want a definition that agrees, for stationary, axisymmetric spacetimes, with Killing vectors of the form $t^\alpha + \Omega\phi^\alpha$, where t^α is the asymptotically timelike Killing vector and ϕ^α the rotational Killing vector. Let χ_t be the family of diffeos generated by k^α, moving each point $P \in M$ a parameter distance t along the integral curve of k^α through P. Although a helical vector is spacelike at distances from the axis larger than $T/2\pi$, its integral curves spiral each period to points that are timelike separated from their starting points; at least they are timelike separated when one is outside a finite region that encloses any horizon or ergosphere. Without this last caveat, one could define a helical vector by the requirement that, for each point P, $\chi_T(P)$ be timelike separated from P. To include the caveat, one requires that the condition hold only outside some sphere. Let $\mathcal{S}$ be a spacelike sphere, and let $\mathcal{T}$ be the timelike surface swept out by the action of χ_t on $\mathcal{S}$: $\mathcal{T}(\mathcal{S}) = \cup_t \chi_t(\mathcal{S})$; we call $\mathcal{T}$ the history of $\mathcal{S}$.

Definition II.1 *A vector field k^α is helical if there is a smallest $T > 0$ for which P and $\chi_T(P)$ are timelike separated for every P outside the history $\mathcal{T}$ of some sphere.*

When the spacetime admits a foliation by timelike lines, this definition is equivalent to the following definition, essentially that of Bonazzola et al. [3]:

Proposition II.1 *A vector field k^α is helical if it can be written in the form*

$$k^\alpha = t^\alpha + \Omega\phi^\alpha, \tag{1}$$

where ϕ^α is spacelike and has circular orbits with parameter length 2π, except where it vanishes; Ω is a constant; and, t^α is timelike outside the history $\mathcal{T}$ of some sphere. Conversely, if a vector field is helical, and if the spacetime can be foliated by timelike curves that respect the action $\chi_T(P)$, then k^α can be written in the form (1). [2]

Because there are spacetimes with helical vectors that do not allow foliations respecting the action $\chi_T(P)$, the Bonazzola et al. definition is slightly more restrictive than ours; they are also more restrictive in requiring the existence of a 2-dimensional submanifold, the axis of symmetry, on which ϕ^α vanishes; and in requiring that t^α be timelike everywhere. Note that, although the proposition displays the intuitive character of a helical vector, t^α and ϕ^α are far from unique. Each foliation of M by a family of timelike curves that respects χ_T gives a different decomposition of k^α of the form (1).

Proof of Proposition. The first part of the Proposition, that a vector of the form (1) is helical, is immediate. We prove as follows that a helical vector can be written in this form. Define a scalar by requiring it to have the value t on $\chi_t(\Sigma)$, with Σ a Cauchy surface. Let t^α be the vector tangent to our congruence of timelike curves, each parametrized by t.

Let ψ_t be the family of diffeos generated by t^α. Each integral curve of k^α can be projected to a circle on Σ by pushing it down to Σ along the timelike congruence: The circle through each point $P \in \Sigma$ is

$$t \to c(t) := \psi_{-t} \circ \chi_t(P).$$

One obtains a circle with parameter length 2π by reparametrizing c, defining $C(s) := c[sT/(2\pi)]$. Finally, define ϕ^α on Σ as the vector field tangent at each point P to the circles $C(t)$ through P; and drag ϕ^α by ψ_t to extend it to M. Then $k^\alpha = t^\alpha + \Omega\phi^\alpha$, with $\Omega = 2\pi/T$. □

In particular a spacetime that is stationary and axisymmetric, with asymptotically timelike Killing vector t^α and rotational Killing vector ϕ^α, has a family of helical Killing vectors $t^\alpha + \Omega\phi^\alpha$, for each Ω. Our primary concern, of course, is with binary systems, spacetimes for which t^α and ϕ^α are not themselves Killing vectors, although, for one value of Ω, $k^\alpha = t^\alpha + \Omega\phi^\alpha$ is.

We have emphasized that spacetimes with a helical Killing vector cannot be asymptotically flat in the exact theory, and a theorem by Gibbons and Stewart [18], showing that $\mathcal{I}$ (null infinity) cannot be periodic, makes this claim precise: No spacetime can have a $\mathcal{I}$ (and hence no spacetime can be asymptotically flat) if it is vacuum outside a compact region and has a helical Killing vector. We can, however, use the Killing vector k^α to define as follows the future and past horizon and the future and past domains of outer communication of a spacetime with a helical Killing vector.

Definition II.2 *A point $x \in M$ is in the future (past) domain of outer communication, $\mathcal{D}^\pm$ if some future-directed (past-directed) timelike curve $c(\lambda)$ through P eventually exits and remains outside the history $\mathcal{T}$ of each sphere S: That is, for each history $\mathcal{T}$ that encloses P, there is some λ_0 for which $c(\lambda)$ is outside of $\mathcal{T}$, all $\lambda > \lambda_0$.*

Definition II.3 *The future (past) event horizon $\mathcal{H}^\pm$ is the boundary of the future (past) domain of outer communication.*

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[2] Without the requirement on the timelike character of t^α, any Killing vector can be written in the form $t^\alpha + \Omega\phi^\alpha$. To restrict *helical* to the vector fields in which we are interested, we had to exclude the spiral Killing vectors of Minkowski space that have the form $s^\alpha + \phi^\alpha$, with s^α a constant *spacelike* vector.

Proposition II.2 *Let the history $\mathcal{T}$ of a spacelike sphere lie in $\mathcal{D}^{\pm}$. Then $\mathcal{H}^{\pm} = \partial I^{\mp}(\mathcal{T})$.*

Proof. Denote by $\mathrm{int}(\mathcal{T})$, the points inside a history $\mathcal{T}$. It suffices to show that $I^-(\mathcal{D}^+) \bigcap \mathrm{int}(\mathcal{T}) = I^-(\mathcal{T}^+) \bigcap \mathrm{int}(\mathcal{T})$. For any $P \in I^-(\mathcal{D}^+) \bigcap \mathrm{int}(\mathcal{T})$, there is a timelike curve from P that exits $\mathcal{T}$ and hence intersects $\mathcal{T}$. Thus $P \in I^-(\mathcal{T}^+) \cap \mathrm{int}(\mathcal{T})$; and, from $\mathcal{T} \subset \mathcal{D}^+ \Rightarrow I^-(\mathcal{T}) \subset I^-(\mathcal{D})$, the result follows. □

The main result of this section is that $\mathcal{H}^{\pm}$ are Killing horizons and hence that they satisfy the zeroth law of black-hole thermodynamics: That is, the Killing vector k^α is tangent on $\mathcal{H}^{\pm}$ to the null generators; and the associated surface gravity κ, defined by

$$k^\beta \nabla_\beta k^\alpha = \kappa k^\alpha, \tag{2}$$

is constant on each connected component of $\mathcal{H}^{\pm}$.

To prove that $\mathcal{H}^{\pm}$ is a Killing horizon (Prop. II.5 below), we will use an analogous theorem proved by Isenberg and Moncrief [19,20] and a strengthened version by Friedrich, Rácz, and Wald [21] (henceforth FR&W), for a class of spacetimes with a compact null surface (see also earlier work by Hawking [22]). Following FR&W, we first show that the spacetime $N, g_{\alpha\beta}$ covers such a compact spacetime. Although our spacetime is not in the class they study, FR&W note that their asymptotic conditions can be relaxed, and we easily extend their proof to spacetimes of the kind considered here.

For convenience in matching our definition and proof to that of FR&W, we consider a subspacetime $N = I^+(M) \bigcap \overline{\mathrm{int}\mathcal{T}}$, for some $\mathcal{T}$ that encloses the fluid and black holes. By choosing a future set, we keep all black holes but discard the bifurcation horizon and white holes that are part of the full spacetime. (To obtain the corresponding results for white holes – for the past horizon – one exchanges future and past). When the surface gravity κ is nonzero, the past-directed null generators reach the bifurcation horizon of M in finite affine parameter length. This means that in N, they are past geodesically incomplete, and that past incompleteness is one of the conditions required for the FR&W proof. The Isenberg-Moncrief version does not require past incompleteness, but does demand that the horizon be analytic. $N, g_{\alpha\beta}$ satisfies the following conditions that define a spacetime of type A$'$.

Definition II.4 *A smooth spacetime $N, g_{\alpha\beta}$ will be said to be of class A$'$ if it has the following properties.*
(i) The spacetime has a Killing vector field k^α that is transverse to a Cauchy surface.[3]
(ii) $N = I^+(N)$
(iii) There is a history $\mathcal{T}$ for which $N = I^+(\mathcal{T})$
(iv) The horizon $\mathcal{H} := \partial I^-(N)$ consists of smooth disconnected components each of which has topology $\mathbb{R} \times S^2$.
(v) The generators of $\mathcal{H}$ are past incomplete. (Alternatively, $\mathcal{H}$ is analytic.)

Proposition II.3 *Let $N, g_{\alpha\beta}$ be a spacetime of type A$'$, satisfying the null energy condition $R^{\alpha\beta} l^\alpha l^\beta \geq 0$, all null l^α. Then on each component of the horizon, there exists a $t_0 \neq 0$ such that χ_{t_0} maps each null geodesic generator of $\mathcal{H}$ to itself.*

We first need to establish for spacetimes of class A$'$ an analog of Prop. 9.3.1 of Hawking and Ellis [23], showing that the shear and divergence of the horizon generators vanish. This implies that the generators are Killing vectors of the horizon, Lie-deriving its degenerate 3-metric.

Lemma II.1 *Let $N, g_{\alpha\beta}$ be a spacetime of class A$'$. On each component of the horizon, the shear and expansion of the null generators vanishes.*

Proof of Lemma II.1. Let S be a Cauchy surface transverse to k^α, $S_t = \chi_t(S)$, and let $\mathcal{B}_t = S_t \bigcap \mathcal{H}$. Because χ_t is an isometry, it maps $\mathcal{H}$ to itself. Then k^α is tangent to $\mathcal{H}$, and the family of slices $\mathcal{F}_t$ foliates $\mathcal{H}$. Because $\mathcal{F}_t$ is mapped to $\mathcal{F}_{t'}$ by the isometry $\chi_{t'-t}$, the area of $\mathcal{F}_t$ is independent of t. This implies that the divergence θ of the horizon's generators vanishes and that the generators have no past endpoints. Finally, using $\theta = 0$ and the null energy condition, the Raychaudhuri equation (optical scalar equation),

$$\frac{d\theta}{d\lambda} = -R_{\alpha\beta} l^\alpha l^\beta - 2\sigma_{\alpha\beta}\sigma^{\alpha\beta} - \frac{1}{2}\theta^2, \tag{3}$$

implies $\sigma_{\alpha\beta} = 0$. □

Proof of Proposition II.3. Once Lemma II.1 is proved, the proof of this proposition is exactly the proof of Prop. 2.1 in FR&W. □

[3] A vector field k^α is *transverse* to a hypersurface S if k^α is nowhere zero on S and nowhere tangent to S.

Definition II.5 *A spacetime $N, g_{\alpha\beta}$ is of class B if it contains a compact orientable, smooth null hypersurface $\mathcal{N}$ that is generated by closed null geodesics.*

(These causality-violating spacetimes are introduced only as part of the proof of Prop. II.4; the spacetimes considered in this paper as models of physical systems are globally hyperbolic.)

Proposition II.4 *Let $N, g_{\alpha\beta}$ be a spacetime of type A'. Then $(int(N), g_{\alpha\beta})$ is a covering spacetime of a spacetime of type B.*

The proof is immediate:

Proof of Prop. II.4 . Because k^α is transverse to a Cauchy surface, χ_t has no fixed points for $t \neq 0$; in particular, for $t = t_0$ of Prop. II.3, χ_{t_0} has no fixed points. Then the factor space $\tilde{N} = \text{int}(N)/\chi_{t_0}$, with induced metric $\tilde{g}_{\alpha\beta}$ has covering spacetime $\text{int}(N), g_{\alpha\beta}$. Because χ_{t_0} maps each generator of $\mathcal{H}$ to itself, $\tilde{\mathcal{H}} = \mathcal{H}/\chi_{t_0}$ is a null hypersurface generated by closed null geodesics. □

Proposition II.5 *In a spacetime of class A' $\mathcal{H}$ is a Killing horizon of k^α. In particular, if, up to a constant scaling, k^α is the only Killing vector in N (or in any subspacetime), then k^α is parallel to the null generators of $\mathcal{H}$.*

Proof. Any neighborhood of a component of the horizon of $(\text{int}(N), g_{\alpha\beta})$ that is disjoint from the fluid covers a vacuum spacetime of type B. Theorem 4.1 of FR&W implies that in a one-sided neighborhood of that component of the horizon, there is a Killing vector $\tilde{K}^\alpha$ normal to the the horizon. The pullback K^α of $\tilde{K}^\alpha$ to the covering space is then a Killing vector on a one-sided neighborhood of the corresponding component of $\mathcal{H}$, normal on $\mathcal{H}$ to $\mathcal{H}$: i.e., $\mathcal{H}$ is a Killing horizon. If each neighborhood has k^α as its only Killing vector (up to an overall scale),then $k^\alpha \propto K^\alpha$ on each component of $\mathcal{H}$, implying that $\mathcal{H}$ is a Killing horizon with Killing vector k^α. □

Corollary (0th Law). The surface gravity κ_i is constant on the ith component of $\mathcal{H}$.

Proof. The proof of the zeroth law of event horizons given in Bardeen et al. [24] establishes the result for any Killing horizon in a spacetime satisfying the null energy condition.

The first law is the content of the next section. The second law, that the area of a black hole cannot decrease, has meaning here only if one extend the definition of event horizon in a way that requires neither a Killing vector nor asymptotic flatness. Black-hole thermodynamics of general spacetimes that are not asymptotically flat has been examined previously [25–28] but the results here appear to be new. [4]

III. FIRST LAW FOR SYSTEMS WITH A SINGLE KILLING VECTOR

We consider spacetimes with black holes and perfect-fluid sources, which have a helical Killing vector or, more generally, a single Killing vector that is transverse to a Cauchy surface and timelike on the support of the fluid. Although such spacetimes will not, in general, be asymptotically flat, one can obtain a generalized first law of thermodynamics in terms of a Noether charge Q associated with the Killing vector field and with an action for the perfect-fluid spacetime. For spacetimes that are asymptotically flat, the overall scaling of a timelike Killing vector is chosen by requiring it to have unit norm at spatial infinity. Here, without asymptotic flatness, the overall scaling cannot be so determined. Instead, in our discussion of the first law, the choice of a family of spacetimes will include the choice of a Killing vector; but readers should keep in mind that nothing in this section restricts the freedom to choose another scaling of the Killing vector for each member of the family of spacetimes.

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[4] In particular, in the isolated horizon framework, for a horizon with a single Killing vector, one shows the existence of a charge E defined on an isolated horizon for which $\delta E = \kappa\delta A$; [27] in our case this is satisfied by the charge $\delta Q_i = \delta Q_{Li} + \delta Q_{Ki}$ defined on the ith disconnected component of the horizon by Eqs. (49) and (53). Our first law, in contrast, relates this change in the black-hole charges to the changes in the Noether charge of a sphere surrounding all black holes and all matter and to the changes in the entropy, baryon number, and circulation of the fluid. The existence of such a first law depends precisely on what is *not* assumed in the isolated horizon framework: a globally defined Killing vector.

We describe a perfect fluid by its four-velocity u^α and stress tensor

$$T^{\alpha\beta} = \epsilon u^\alpha u^\beta + p q^{\alpha\beta}, \tag{4}$$

where p is the fluid's pressure, ϵ its energy density, and

$$q^{\alpha\beta} = g^{\alpha\beta} + u^\alpha u^\beta \tag{5}$$

is the projection orthogonal to u^α. We assume that the fluid satisfies an equation of state of the form

$$p = p(\rho, s), \ \ \epsilon = \epsilon(\rho, s), \tag{6}$$

with ρ the baryon-mass density and s the entropy per unit baryon mass. (That is, $\rho := m_B n$, with n the number density of baryons and m_B the average baryon mass.)

Given a family of perfect-fluid spacetimes specified by

$$\mathcal{Q}(\lambda) := [g_{\alpha\beta}(\lambda), u^\alpha(\lambda), \rho(\lambda), s(\lambda)], \tag{7}$$

one defines the eulerian change in each quantity by $\delta\mathcal{Q} := \frac{d}{d\lambda}\mathcal{Q}(\lambda)$.

We introduce a lagrangian displacement ξ^α in the following way: Let $\mathcal{Q} := \mathcal{Q}(\lambda)$, and let Ψ_λ be a diffeo mapping each trajectory (worldline) of the initial fluid to a corresponding trajectory of the configuration $\mathcal{Q}(\lambda)$. Then the tangent $\xi^\alpha(P)$ to the path $\lambda \to \Psi_\lambda(P)$ can be regarded as a vector joining the fluid element at P in the configuration $\mathcal{Q}(\lambda)$ to a fluid element in a nearby configuration. The lagrangian change in a quantity at $\lambda = 0$,[5] is then given by

$$\Delta\mathcal{Q} := \frac{d}{d\lambda}\Psi_{-\lambda}\mathcal{Q}(\lambda)|_{\lambda=0} = (\delta + \pounds_\xi)\mathcal{Q}. \tag{8}$$

The first law will be written in terms of integrals over a spacelike hypersurface Σ, transverse to k^α, whose boundary

$$\partial\Sigma = S \cup_i \mathcal{B}_i, \tag{9}$$

is the union of black hole boundaries $\mathcal{B}_i$ ($\mathcal{B}_i$, is the ith disconnected component of $\Sigma \cap \mathcal{H}^+$), and a 2-sphere S that encloses the fluid and all black holes. Define a scalar t by setting $t = 0$ on Σ and requiring $k^\alpha\nabla_\alpha t = 1$.

We can write u^α in the form,

$$u^\alpha = u^t(k^\alpha + v^\alpha), \tag{10}$$

with $u^t := u^\alpha\nabla_\alpha t$ and v^α a vector field on Σ,

$$v^\alpha\nabla_\alpha t = 0. \tag{11}$$

The fact that Ψ_λ maps fluid trajectories to fluid trajectories and the normalization $u^\alpha u_\alpha = -1$ imply [29–31]

$$\Delta u^\alpha = \frac{1}{2}u^\alpha u^\beta u^\gamma \Delta g_{\beta\gamma}. \tag{12}$$

One obtains an action for a perfect-fluid spacetime by considering perturbations for which the entropy and baryon mass of each fluid element are conserved; and we use this action to define a Noether charge Q associated with k^α, for each spacetime $\mathcal{Q}(\lambda)$. Then for general perturbations, in which the entropy and baryon mass of each fluid element are unconstrained, we use the charge Q to write a form of the first law for perfect-fluid spacetimes that have one Killing vector and a Killing horizon (and that are not, in general, asymptotically flat).

When the entropy and baryon mass of each fluid element are conserved along the family $\mathcal{Q}(\lambda)$, we have

$$\Delta s = 0 \qquad \text{and} \qquad \Delta(\rho u^\alpha \sqrt{-g}) = 0, \tag{13}$$

implying

[5] The lagrangian change is analogously defined at any λ_0: The diffeo $\widetilde{\Psi}_\lambda = \Psi_{\lambda+\lambda_0}\Psi_{\lambda_0}^{-1}$ maps each fluid trajectory in the configuration $\mathcal{Q}(\lambda_0)$ to the corresponding trajectory of $\mathcal{Q}(\lambda+\lambda_0)$, whence $\Delta\mathcal{Q}(\lambda_0) := \frac{d}{d\lambda}\widetilde{\Psi}_{-\lambda}\mathcal{Q}(\lambda+\lambda_0)|_{\lambda=0}$.

$$\frac{\Delta\rho}{\rho} = -\frac{1}{2}q^{\alpha\beta}\Delta g_{\alpha\beta}; \tag{14}$$

and the local first law of thermodynamics for the fluid,

$$\Delta\epsilon = \rho T\Delta s + h\Delta\rho, \tag{15}$$

with

$$h = \frac{\epsilon + p}{\rho}, \tag{16}$$

yields

$$\frac{\Delta\epsilon}{\epsilon + p} = \frac{\Delta\rho}{\rho} = -\frac{1}{2}q^{\alpha\beta}\Delta g_{\alpha\beta}. \tag{17}$$

From these relations, it follows that the scalar density

$$\mathcal{L} = \left(\frac{1}{16\pi}R - \epsilon\right)\sqrt{-g} \tag{18}$$

is a lagrangian density for a perfect fluid space time. That is,

$$\delta\mathcal{L} = \frac{1}{16\pi}\delta(R\sqrt{-g}) - \Delta(\epsilon\sqrt{-g}) + \nabla_\alpha(\epsilon\xi^\alpha)\sqrt{-g}, \tag{19}$$

and (when $\Delta s = 0$ and $\Delta(\rho u^\alpha\sqrt{-g}) = 0$), we have,

$$\begin{aligned}
\frac{1}{\sqrt{-g}}\Delta(\epsilon\sqrt{-g}) &= -\frac{1}{2}(\epsilon + p)q^{\alpha\beta}\Delta g_{\alpha\beta} + \frac{1}{2}\epsilon g^{\alpha\beta}\Delta g_{\alpha\beta} \\
&= -\frac{1}{2}T^{\alpha\beta}\Delta g_{\alpha\beta} \\
&= -\frac{1}{2}T^{\alpha\beta}\delta g_{\alpha\beta} + \xi_\alpha\nabla_\beta T^{\alpha\beta} - \nabla_\alpha(T^{\alpha\beta}\xi_\beta).
\end{aligned} \tag{20}$$

That $\mathcal{L}$ is a lagrangian density is then expressed by the equation [32]

$$\frac{1}{\sqrt{-g}}\delta\mathcal{L} = -\frac{1}{16\pi}(G^{\alpha\beta} - 8\pi T^{\alpha\beta})\delta g_{\alpha\beta} - \xi_\alpha\nabla_\beta T^{\alpha\beta} + \nabla_\alpha\Theta^\alpha, \tag{21}$$

with

$$\Theta^\alpha = (\epsilon + p)q^{\alpha\beta}\xi_\beta + \frac{1}{16\pi}(g^{\alpha\gamma}g^{\beta\delta} - g^{\alpha\beta}g^{\gamma\delta})\nabla_\beta\delta g_{\gamma\delta}. \tag{22}$$

Now one can associate with $\mathcal{L}$ a family of Noether charges [13,10–12,14,15] [6]

$$Q = \oint_S Q^{\alpha\beta}dS_{\alpha\beta}, \tag{23}$$

where

$$Q^{\alpha\beta} = -\frac{1}{8\pi}\nabla^\alpha k^\beta + k^\alpha B^\beta - k^\beta B^\alpha, \tag{24}$$

and $B^\alpha(\lambda)$ is any family of vector fields that satisfies

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[6]Our Noether formalism is similar to Iyer's extension of the Wald-Iyer work to perfect fluid spacetimes. [12]. Like Schutz and Sorkin, [13,14], however, we use vectors instead of forms, and our lagrangian displacement arises from a map Ψ_λ from the manifold to itself, not, as in Iyer, from a projection onto the manifold of fluid trajectories.

$$\frac{1}{\sqrt{-g}}\frac{d}{d\lambda}(B^\alpha\sqrt{-g}) = \Theta^\alpha. \tag{25}$$

By choosing $B^\alpha(0) = 0$, we make $Q(\lambda)$ finite; and, as we will see, Q is *independent of the sphere S*, as long as S encloses the fluid and any black holes. Outside the matter,

$$B^\alpha = \frac{1}{16\pi}(g^{\alpha\gamma}g^{\beta\delta} - g^{\alpha\beta}g^{\gamma\delta})|_{\lambda=0}\overset{\circ}{\nabla}_\beta g_{\gamma\delta}(\lambda) + \mathrm{O}(\lambda^2), \tag{26}$$

where, $\overset{\circ}{\nabla}_\beta$ is the covariant derivative of the metric $g_{\alpha\beta}(0)$.

The generalized first law will be found by evaluating the change δQ in this Noether charge, allowing perturbations that change the baryon number and entropy of each fluid element. We restrict the gauge in two ways: We use the diffeomorphism gauge freedom to set $\delta k^\alpha = 0$. The description of fluid perturbations in terms of a lagrangian displacement ξ^α has a second kind of gauge freedom: a class of trivial displacements, including all displacements of the form fu^α, yield no eulerian change in the fluid variables. We use this freedom to set $\Delta t = 0$. Because $\delta t = 0$ (t is not dynamical), this is equivalent to the condition $\xi^t = 0$. Eq. (12) now implies

$$\frac{\Delta u^t}{u^t} = \frac{1}{2}u^\alpha u^\beta \Delta g_{\alpha\beta}. \tag{27}$$

Then, from Eqs. (12) and (27), we have $\Delta u^\alpha = \Delta u^t(k^\alpha + v^\alpha)$, while, by Eq. (10), $\Delta u^\alpha = \Delta[u^t(k^\alpha + v^\alpha)]$; thus

$$\Delta(k^\alpha + v^\alpha) = 0. \tag{28}$$

For perturbations that include changes in baryon number and entropy, Eqs. (14) and (20) are replaced by

$$\frac{\Delta\rho}{\rho} = -\frac{1}{2}q^{\alpha\beta}\Delta g_{\alpha\beta} + \frac{\Delta(\rho\sqrt{-g}u^t)}{\rho\sqrt{-g}u^t}, \tag{29}$$

and

$$\frac{1}{\sqrt{-g}}\Delta(\epsilon\sqrt{-g}) = \rho T\Delta s + \frac{h}{u^t\sqrt{-g}}\Delta(\rho u^t\sqrt{-g}) - \frac{1}{2}T^{\alpha\beta}\delta g_{\alpha\beta} + \xi_\alpha\nabla_\beta T^{\alpha\beta} - \nabla_\alpha(T^{\alpha\beta}\xi_\beta); \tag{30}$$

and the change in the lagrangian density becomes,

$$\frac{1}{\sqrt{-g}}\delta\mathcal{L} = -\rho T\Delta s - \frac{h}{u^t\sqrt{-g}}\Delta(\rho u^t\sqrt{-g}) - \frac{1}{16\pi}(G^{\alpha\beta} - 8\pi T^{\alpha\beta})\delta g_{\alpha\beta} - \xi_\alpha\nabla_\beta T^{\alpha\beta} + \nabla_\alpha\Theta^\alpha. \tag{31}$$

To find the change δQ in the Noether charge, we first compute the difference,

$$\delta[Q - \sum_i Q_i], \tag{32}$$

between the charge on the sphere S and the sum of the charges on the black holes $\mathcal{B}_i$. As we show below, *this quantity is invariant under gauge transformations that respect the Killing symmetry.* Write $Q = Q_K + Q_L$ (Q_K the Komar charge, Q_L an additional contribution involving the lagrangian density), with

$$Q_K = -\frac{1}{8\pi}\oint_S \nabla^\alpha k^\beta dS_{\alpha\beta}, \qquad \delta Q_L = \oint_S (k^\alpha\Theta^\beta - k^\beta\Theta^\alpha)dS_{\alpha\beta}, \tag{33}$$

$$\delta(Q - \sum_i Q_i) = \delta(Q_K - \sum_i Q_{Ki}) + \delta(Q_L - \sum_i Q_{Li}). \tag{34}$$

From the identity

$$\nabla_\beta\nabla^\alpha k^\beta = R^\alpha{}_\beta k^\beta, \tag{35}$$

we have

$$Q_K - \sum_i Q_{Ki} = -\frac{1}{8\pi}\oint_{\partial\Sigma} \nabla^\alpha k^\beta dS_{\alpha\beta} = -\frac{1}{8\pi}\int_\Sigma R^\alpha{}_\beta k^\beta dS_\alpha \tag{36}$$

$$= -\frac{1}{8\pi}\int_\Sigma G^\alpha_\beta k^\beta dS_\alpha - \frac{1}{16\pi}\int_\Sigma R k^\alpha dS_\alpha. \tag{37}$$

Now

$$\begin{aligned} -T^\alpha{}_\beta k^\beta dS_\alpha &= -T^\alpha{}_\beta (k^\beta + v^\beta) dS_\alpha + T^\alpha{}_\beta v^\beta dS_\alpha \\ &= \epsilon\, k^\alpha dS_\alpha + (\epsilon + p) u^\alpha u_\beta v^\beta dS_\alpha, \end{aligned} \tag{38}$$

whence

$$Q_K - \sum_i Q_{Ki} = -\int_\Sigma \left(\frac{1}{16\pi}R - \epsilon\right) k^\alpha dS_\alpha + \int_\Sigma (\epsilon+p) u^\alpha u_\beta v^\beta dS_\alpha - \int_\Sigma \frac{1}{8\pi}(G^\alpha{}_\beta - 8\pi T^\alpha{}_\beta) k^\beta dS_\alpha. \tag{39}$$

and

$$\begin{aligned} \delta(Q_K - \sum_i Q_{Ki}) = &-\int_\Sigma \delta\mathcal{L}\, d^3x + \int_\Sigma \Delta\left[(\epsilon+p) u^\alpha u_\beta v^\beta dS_\alpha\right] \\ &-\delta\int_\Sigma \frac{1}{8\pi}(G^\alpha{}_\beta - 8\pi T^\alpha{}_\beta) k^\beta dS_\alpha. \end{aligned} \tag{40}$$

The second term on the right of Eq. (34) is given by

$$\begin{aligned} \delta(Q_L - \sum_i Q_{Li}) &= \oint_{\partial\Sigma} (k^\alpha \Theta^\beta - k^\beta \Theta^\alpha) dS_{\alpha\beta} \\ &= \int_\Sigma \nabla_\beta (k^\alpha \Theta^\beta - k^\beta \Theta^\alpha) dS_\alpha \\ &= \int_\Sigma \nabla_\beta \Theta^\beta k^\alpha dS_\alpha - \int_\Sigma \pounds_k \Theta^\alpha dS_\alpha, \end{aligned} \tag{41}$$

where we have used the relation $\nabla_\alpha k^\alpha = 0$ to obtain the last equality, and $\nabla_\beta \Theta^\beta$ is given by Eq. (31). Then, adding Eqs. (40) and (41), and using the relations

$$\Delta\left[(\epsilon+p) u^\alpha u_\beta v^\beta dS_\alpha\right] = h u_\beta v^\beta \Delta(\rho u^\alpha dS_\alpha) + v^\beta \Delta(h u_\beta) \rho u^\alpha dS_\alpha + (\epsilon+p) u^\alpha u_\beta\, \pounds_k \xi^\beta dS_\alpha, \tag{42}$$

where $\Delta v^\beta = -\Delta k^\beta = \pounds_k \xi^\beta$ is used and

$$\pounds_k \Theta^\alpha dS_\alpha = (\epsilon+p) q^\alpha{}_\beta\, \pounds_k \xi^\beta dS_\alpha = (\epsilon+p) u^\alpha u_\beta\, \pounds_k \xi^\beta dS_\alpha, \tag{43}$$

where $\pounds_k \xi^\beta\, \nabla_\beta t = 0$ is used, we obtain an expression for $\delta(Q - \sum_i Q_i)$:

$$\begin{aligned} \delta(Q - \sum_i Q_i) = &\int_\Sigma \left[\frac{T}{u^t}\Delta s\ \rho u^\alpha dS_\alpha + \left(\frac{h}{u^t} + h u_\beta v^\beta\right)\Delta(\rho u^\alpha dS_\alpha) + v^\beta \Delta(h u_\beta) \rho u^\alpha dS_\alpha\right] \\ &-\frac{1}{8\pi}\delta\int_\Sigma (G^\alpha{}_\beta - 8\pi T^\alpha{}_\beta) k^\beta dS_\alpha + \int_\Sigma \left[\frac{1}{16\pi}(G^{\alpha\beta} - 8\pi T^{\alpha\beta})\delta g_{\alpha\beta} + \xi^\beta \nabla_\alpha T^\alpha_\beta\right] k^\gamma dS_\gamma. \end{aligned} \tag{44}$$

We next evaluate the black-hole charges Q_i. Recall that, by Prop. (II.5), k^α is tangent, on each disconnected component $\mathcal{H}_i$, to the null generators of the horizon, with surface gravity κ_i given by

$$k^\beta \nabla_\beta k^\alpha = \kappa_i k^\alpha. \tag{45}$$

On each $\mathcal{B}_i$, let $\mathfrak{n}^\alpha$ be the unique null vector field orthogonal to $\mathcal{B}_i$ and satisfying $\mathfrak{n}_\alpha k^\alpha = -1$. The area element of $\mathcal{B}_i$ is then

$$dS_{\alpha\beta} = \frac{1}{2}(k_\alpha \mathfrak{n}_\beta - k_\beta \mathfrak{n}_\alpha) dA. \tag{46}$$

Using the Killing equation, $\nabla^\alpha k^\beta = \nabla^{[\alpha} k^{\beta]}$, and Eq. (2) to evaluate the integrand of Q_{Ki}, we have

$$\nabla^\alpha k^\beta \frac{1}{2}(k_\alpha \mathfrak{n}_\beta - k_\beta \mathfrak{n}_\alpha) = k^\beta \nabla_\beta k^\alpha \mathfrak{n}_\alpha = -\kappa_i, \tag{47}$$

implying

$$Q_{Ki} = -\frac{1}{8\pi}\oint_{\mathcal{B}_i} \nabla^\alpha k^\beta dS_{\alpha\beta} = \frac{1}{8\pi}\kappa_i A_i. \tag{48}$$

Finally, following Bardeen et al. [24], we show that

$$\delta Q_{Li} = -\frac{1}{8\pi}\delta\kappa_i A_i. \tag{49}$$

Using $\delta(\nabla_\alpha k_\beta) = \delta(\nabla_{[\alpha} k_{\beta]}) = \nabla_{[\alpha}\delta k_{\beta]}$, we have

$$\begin{aligned}\delta\kappa_i &= \delta(\mathfrak{n}^\alpha k^\beta \nabla_\alpha k_\beta) \\ &= \delta\mathfrak{n}^\alpha k^\beta \nabla_\alpha k_\beta + \mathfrak{n}^\alpha k^\beta \nabla_{[\alpha}\delta k_{\beta]}.\end{aligned} \tag{50}$$

Because the horizon is unchanged in our gauge, and k_α is parallel to the null normal to $\mathcal{H}_i$, $\delta k_\alpha = a k_\alpha$, some function a on $\mathcal{H}_i$. Then

$$\begin{aligned}\delta\mathfrak{n}^\alpha k^\beta \nabla_\alpha k_\beta &= -\delta\mathfrak{n}^\alpha \kappa_i k_\alpha = \kappa_i \mathfrak{n}^\alpha \delta k_\alpha = -\kappa_i a \\ &= -a\mathfrak{n}^\alpha k_\beta \nabla_\alpha k^\beta = -\mathfrak{n}^\alpha \delta k_\beta \nabla_\alpha k^\beta \\ &= k^\alpha \mathfrak{n}^\beta \nabla_\alpha \delta k_\beta,\end{aligned} \tag{51}$$

where, in the last equality, we have used $\pounds_k \delta k_\alpha = 0$. From Eqs. (50) and (51) and from the vanishing of $\delta\sigma_{\alpha\beta}$ and $\delta\theta$, we have

$$\begin{aligned}\delta\kappa_i &= \frac{1}{2}(k^\alpha \mathfrak{n}^\beta + k^\beta \mathfrak{n}^\alpha)\nabla_\alpha \delta k_\beta = -\frac{1}{2}\nabla^\alpha \delta k_\alpha \\ &= -\frac{1}{2}k^\alpha \nabla^\beta \delta g_{\alpha\beta}.\end{aligned} \tag{52}$$

Now

$$\begin{aligned}\delta Q_{Li} &= \oint_{\mathcal{B}_i}(k^\alpha \Theta^\beta - k^\beta \Theta^\alpha) dS_{\alpha\beta} \\ &= \frac{1}{8\pi}\oint_{\mathcal{B}_i} k^\alpha (g^{\beta\delta} g^{\gamma\epsilon} - g^{\beta\gamma} g^{\delta\epsilon})\nabla_\gamma \delta g_{\delta\epsilon}\,\frac{1}{2}(k_\alpha \mathfrak{n}_\beta - k_\beta \mathfrak{n}_\alpha) dA \\ &= \frac{1}{16\pi}\oint_{\mathcal{B}_i} k^\alpha \nabla^\beta \delta g_{\alpha\beta} dA \\ &= -\frac{1}{8\pi}\delta\kappa_i A_i\end{aligned} \tag{53}$$

The first law now follows from Eq. (44) for $\delta(Q - \sum_i Q_i)$, Eq. (48) for Q_{Ki}, and Eq. (53) for $\tilde{Q}_{Li}$:

$$\begin{aligned}\delta Q = &\int_\Sigma \left[\rho \frac{T}{u^t}\Delta s u^\alpha dS_\alpha + \left(\frac{h}{u^t} + h u_\beta v^\beta\right)\Delta(\rho u^\alpha dS_\alpha) + v^\beta \Delta(h u_\beta)\rho u^\alpha dS_\alpha\right] + \sum_i \frac{1}{8\pi}\kappa_i \delta A_i \\ &-\frac{1}{8\pi}\delta\int_\Sigma (G^\alpha{}_\beta - 8\pi T^\alpha{}_\beta) k^\beta dS_\alpha + \int_\Sigma \left[\frac{1}{16\pi}(G^{\alpha\beta} - 8\pi T^{\alpha\beta})\delta g_{\alpha\beta} + \xi^\beta \nabla_\alpha T^\alpha{}_\beta\right] k^\gamma dS_\gamma.\end{aligned} \tag{54}$$

When the family of spacetimes satisfies the field equations, the last line vanishes and we obtain a first law of thermodynamics in the form

$$\delta Q = \int_\Sigma \left[\rho \frac{T}{u^t}\Delta s u^\alpha dS_\alpha + \left(\frac{h}{u^t} + h u_\beta v^\beta\right)\Delta(\rho u^\alpha dS_\alpha) + v^\beta \Delta(h u_\beta)\rho u^\alpha dS_\alpha\right] + \sum_i \frac{1}{8\pi}\kappa_i \delta A_i. \tag{55}$$

Equivalently, writing

$$\bar{T} := \frac{T}{u^t}, \qquad \bar{\mu} := \frac{\mu}{u^t m_{\rm B}} = \frac{h - Ts}{u^t}, \tag{56}$$

and

$$dM_{\rm B} := \rho u^\alpha dS_\alpha, \qquad dS := s\ dM_{\rm B}, \qquad dC_\alpha := h u_\alpha dM_{\rm B}, \tag{57}$$

we have

$$\delta Q = \int_\Sigma \left[\bar{T}\Delta dS + \bar{\mu}\Delta dM_{\rm B} + v^\alpha \Delta dC_\alpha\right] + \sum_i \frac{1}{8\pi}\kappa_i \delta A_i. \tag{58}$$

The relation between this form and that for an asymptotically flat spacetime with two Killing vectors, t^α and ϕ^α, will be found in Sect. IV B.

We noted above that the difference $\delta(Q - Q_i)$ is gauge invariant. In fact, we can see as follows that $\delta(Q_K - \sum_i Q_{Ki})$ and $\delta(Q_L - \sum_i Q_{Li})$ are separately invariant under gauge transformations that respect the symmetry k^α. The gauge transformation associated with a vector field η^α is given by

$$\delta_\eta \mathcal{Q} = \pounds_\eta \mathcal{Q}, \qquad \xi^\alpha(\eta) = -\eta^\alpha. \tag{59}$$

The corresponding lagrangian change in any quantity is then identically zero:

$$\Delta_\eta = \delta_\eta + \pounds_{-\eta} = 0. \tag{60}$$

From Eq. (41), (31), and (60) the change in $\delta(Q_L - \sum_i Q_{Li})$ due to a gauge transformation is given by

$$\delta(Q_L - \sum_i Q_{Li}) = \int_\Sigma \nabla_\beta \Theta^\beta k^\alpha dS_\alpha = \int_\Sigma \delta_\eta \mathcal{L}\ d^3x, \tag{61}$$

when the field equations are satisfied. Decomposing η in the manner

$$\eta^\alpha = \eta^\beta \nabla_\beta t\, k^\alpha + \hat{\eta}^\alpha, \tag{62}$$

with $\hat{\eta}^\alpha \nabla_\alpha t = 0$, and using $\pounds_k \eta^\alpha = 0$, we have $\delta_\eta \mathcal{L} = \pounds_\eta \mathcal{L} = \nabla_\alpha(\mathcal{L}\hat{\eta}^\alpha)$,

$$\delta(Q_L - \sum_i Q_{Li}) = \int_\Sigma \partial_\alpha(\mathcal{L}\hat{\eta}^\alpha) d^3x = 0, \tag{63}$$

because $\mathcal{L}$ vanishes outside the fluid (on $\partial\Sigma$).

Similarly, from Eq. (37),

$$\delta(Q_K - \sum_i Q_{Ki}) = -\frac{1}{8\pi}\delta \int_\Sigma R^\alpha{}_\beta k^\beta dS_\alpha. \tag{64}$$

Again, for a gauge transformation that respect the Killing symmetry, the right side is an integral over the boundary $\partial\Sigma$ of a quantity that vanishes outside the fluid.

Lastly, we verify the asertion made previously, that Q is independent of the 2-surface S on which it is evaluated, if S encloses the fluid and any black holes. This is immediate for Q_K from Eq. (37) and (48). For Q (and Q_L), it follows from the fact that $Q = Q_K$ at $\lambda = 0$, together with the implication of Eq. (54) that $\frac{dQ}{d\lambda} = \delta Q$ is independent of S along any sequence of equilibria $\mathcal{Q}(\lambda)$.

A. First law in Hamiltonian framework

In applying the first law to spacetimes that are spatially conformally flat, we will need to write it in a 3+1 form, with metric γ_{ab} on Σ and its conjugate momentum π^{ab} as independent variables. Until Eq. (79) of this section, the vector field k^α that generates time evolution is not assumed to be a Killing vector.

Let $\Sigma = \Sigma_0$ be a Cauchy surface transverse to k^α, and let $\Sigma_t = \chi_t(\Sigma)$, with χ_t the family of diffeos generated by k^α. Denote by $\gamma_{ab}(t)$ the spatial metric on Σ_t. Let n^α be the future-pointing unit normal to this foliation, and recall that one can identify spatial tensors on Σ_t with spacetime tensors that are orthogonal on all of their indices to n^α. In particular, the projection $\gamma_{\alpha\beta}$ orthogonal to n_α,

$$\gamma_{\alpha\beta} = g_{\alpha\beta} + n_\alpha n_\beta, \tag{65}$$

is the 4-tensor associated with the family of 3-metrics $\gamma_{ab}(t)$ on the slices Σ_t. Although k^α is not everywhere timelike, the fact that it is transverse to a family of spacelike hypersurfaces means that we can introduce a nonvanishing lapse α and a shift ω^α that relate $\partial_t \equiv k^\alpha$ to n^α in the usual way,

$$k^\alpha = \alpha n^\alpha + \omega^\alpha, \qquad \omega^\alpha n_\alpha = 0. \tag{66}$$

Then, in a chart $\{t, x^i\}$ for which Σ_t is a t =constant surface, the metric $g_{\alpha\beta} = \gamma_{\alpha\beta} - n_\alpha n_\beta$ has the form

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \omega^i dt)(dx^j + \omega^j dt). \tag{67}$$

With D_a the covariant derivative of the spatial metric γ_{ab}, the extrinsic curvature of Σ_t is given by

$$K_{ab} = -\frac{1}{2}\pounds_n \gamma_{ab} = \frac{1}{2\alpha}(-\dot\gamma_{ab} + D_a\omega_b + D_b\omega_a), \tag{68}$$

where $\dot\gamma_{ab}$ is the pullback to Σ of $\pounds_k \gamma_{\alpha\beta}$, vanishing when k^α is a Killing vector.

By taking as independent variables the quantities $\pi^{ab}, \gamma_{ab}, \alpha,$, with

$$\pi^{ab} = -(K^{ab} - \gamma^{ab}K)\gamma^{1/2}, \tag{69}$$

we now generalize the derivation of the first law to permit independent variations of $\delta\pi^{ab}, \delta\gamma_{ab}, \delta\alpha, \delta\omega^a$.

In terms of Hamiltonian metric variables, the gravitational lagrangian density takes the form [33]

$$R\sqrt{-g} = \pi^{ab}\dot\gamma_{ab} - \alpha\mathcal{H}_G - \omega_a\mathcal{C}^a_G + D_a(-2D^a\alpha\gamma^{\frac{1}{2}} - 2\omega^b\pi^a{}_b + \omega^a\pi) - \dot\pi, \tag{70}$$

where

$$\mathcal{H}_G := -2G^{\alpha\beta}n_\alpha n_\beta\,\gamma^{\frac{1}{2}} = -\,{}^3R\gamma^{\frac{1}{2}} + (\pi_{ab}\pi^{ab} - \frac{1}{2}\pi^2)\gamma^{-\frac{1}{2}}, \tag{71}$$

$$\mathcal{C}^a_G := -2G^{\alpha\beta}\gamma_\alpha{}^a n_\beta\,\gamma^{\frac{1}{2}} = -2D_b\pi^{ab}. \tag{72}$$

Regarding $\mathcal{L} = (\frac{1}{16\pi}R - \epsilon)\sqrt{-g}$ as a function of $\pi^{ab}, \gamma_{ab}, \alpha, \omega^a$ and the fluid variables, we rewrite Eq. (31) in the manner

$$\begin{aligned}\delta\mathcal{L} = &-\alpha\gamma^{\frac{1}{2}}\rho T\Delta s - \frac{h}{u^t}\Delta(\rho u^t\,\alpha\gamma^{\frac{1}{2}})\\ &+\frac{1}{16\pi}\{-\delta\alpha\mathcal{H} - \delta\omega^a\mathcal{C}_a + \delta\pi^{ab}[\dot\gamma_{ab} - D_a\omega_b - D_b\omega_a - 2\alpha(\pi_{ab} - \frac{1}{2}\gamma_{ab}\pi)\gamma^{-\frac{1}{2}}]\\ &\qquad - \delta\gamma_{ab}(G^{ab} - 8\pi S^{ab})\alpha\gamma^{\frac{1}{2}}\} - \xi_\alpha\nabla_\beta T^{\alpha\beta}\alpha\gamma^{\frac{1}{2}} + D_a\tilde\Theta^a\gamma^{\frac{1}{2}} - \frac{1}{16\pi}(\delta\pi^{ab}\,\gamma_{ab})^\bullet\ .\end{aligned} \tag{73}$$

Here, denoting the pullback to Σ of σ_α by $\gamma^\alpha_a\sigma_\alpha$, we have set

$$\rho_H := T_{\alpha\beta}n^\alpha n^\beta,\quad j_a := -T_{\alpha\beta}\gamma^\alpha_a n^\beta,\quad S_{ab} := T_{\alpha\beta}\gamma^\alpha_a\gamma^\beta_b, \tag{74}$$

$$\mathcal{H} := \mathcal{H}_G + 16\pi\rho_H\gamma^{\frac{1}{2}}, \qquad \mathcal{C}^a := \mathcal{C}^a_G - 16\pi j^a\gamma^{\frac{1}{2}}; \tag{75}$$

and the remaining quantities in the last line of (73) are given in terms of $(\pi^{ab}, \gamma_{ab}, \alpha, \omega^a)$ by

$$\begin{aligned}\tilde\Theta^a = \frac{1}{16\pi}\{&[-2\delta(D^a\alpha\,\gamma^{1/2}) + (\omega^a\gamma_{bc}\delta\pi^{bc} + \pi\delta\omega^a - 2\pi^a{}_b\delta\omega^b)]\gamma^{-\frac{1}{2}}\\ &+ (\gamma^{ac}\gamma^{bd} - \gamma^{ab}\gamma^{cd})(\alpha D_b\delta\gamma_{cd} - D_b\alpha\,\delta\gamma_{cd})\\ &+ \alpha(\epsilon + p)q^a{}_b\xi^b - \alpha\omega^a j^b\xi_b\},\end{aligned} \tag{76}$$

$$G^{ab} = \dot{\pi}^{ab}\alpha^{-1}\gamma^{-\frac{1}{2}} + {}^3R^{ab} - \frac{1}{2}\gamma^{ab}\,{}^3R + (2\pi^{ac}\pi^b{}_c - \pi\pi^{ab} - \frac{1}{2}\gamma^{ab}\pi^{cd}\pi_{cd} + \frac{1}{4}\gamma^{ab}\pi^2)\gamma^{-1}$$
$$- \frac{1}{\alpha}(D^aD^b\alpha - \gamma^{ab}D^2\alpha) + \frac{2}{\alpha}\pi^{c(b}D_c\omega^{a)}\gamma^{-\frac{1}{2}} - \frac{1}{\alpha}D_c(\pi^{ab}\omega^c)\gamma^{-\frac{1}{2}}, \tag{77}$$

and

$$\alpha\xi_\beta\nabla_\alpha T^{\alpha\beta} = \xi_b[D_\alpha(\alpha T^{ab}) + D^b\alpha\,\rho_H - j_aD^b\omega^a - D_a(\omega^a j^b)]. \tag{78}$$

For k^α a Killing vector and A^α any vector field Lie derived by k^α, we have the identities

$$\nabla_\alpha A^\alpha\sqrt{-g} = D_a\tilde{A}^a\sqrt{\gamma}, \tag{79}$$

$$\int_{\partial\Sigma}(k^\alpha A^\beta - k^\beta A^\alpha)dS_{\alpha\beta} = \int_{\partial\Sigma}\tilde{A}^a dS_a, \tag{80}$$

where

$$\tilde{A}_a = \alpha A_\alpha\gamma^\alpha_a + \omega_a A_\alpha n^\alpha, \tag{81}$$

and dS_a is along the outward normal to $\partial\Sigma$ in Σ. In particular, the vector $\tilde{\Theta}_a$ of Eq. (76) is related to Θ^α by $\tilde{\Theta}_a = \alpha\Theta_\alpha\gamma^\alpha_a + \omega_a\Theta_\alpha n^\alpha$, implying

$$\delta Q_L = \int_S \tilde{\Theta}^a dS_a, \quad \delta Q_{Li} = -\int_{\mathcal{B}_i}\tilde{\Theta}^a dS_a. \tag{82}$$

Q_K can be expressed in terms of $(\pi^{ab}, \gamma_{ab}, \alpha, \omega^a)$ by writing

$$\nabla_\alpha k_\beta\gamma^\alpha_a n^\beta = \gamma^\alpha_a\nabla_\alpha(k_\beta n^\beta) - \gamma^\alpha_a\nabla_\alpha n_\beta\,k^\beta$$
$$= -D_a\alpha + K_{ab}\omega^b, \tag{83}$$

with $K_{ab} = -(\pi_{ab} - \frac{1}{2}\gamma_{ab}\pi)\gamma^{-\frac{1}{2}}$. Then

$$Q_K - \sum_i Q_{Ki} = \frac{1}{8\pi}\int_{\partial\Sigma}(D^a\alpha - K^a{}_b\omega^b)dS_a$$
$$= -\frac{1}{8\pi}\int_\Sigma R^\alpha{}_\beta k^\beta dS_\alpha, \tag{84}$$

with [34]

$$R^\alpha{}_\beta k^\beta n_\alpha|_\Sigma = D_a(D^a\alpha - K^a{}_b\omega^b). \tag{85}$$

We can verify directly that $R^\alpha{}_\beta k^\beta n_\alpha$ takes the form (85), when written in Hamiltonian variables, using the Hamiltonian forms already given for $\mathcal{H}_G$, $\mathcal{C}_{Ga}$, and G^{ab}. Eq. (66) implies

$$R^\alpha{}_\beta k^\beta n_\alpha\big|_\Sigma = -\frac{1}{4}\alpha\gamma^{-\frac{1}{2}}\mathcal{H}_G - \frac{1}{2}\gamma^{-\frac{1}{2}}\mathcal{C}_{Ga}\omega^a + \frac{1}{2}\alpha\gamma_{ab}G^{ab}. \tag{86}$$

Eq. (77) gives

$$\gamma_{ab}G^{ab} = -\frac{1}{2}{}^{(3)}R + \frac{2}{\alpha}D^2\alpha + \gamma^{-1}(\frac{1}{2}\pi^{ab}\pi_{ab} - \frac{1}{4}\pi^2) + \frac{2}{\alpha}\pi^{ab}D_a\omega_b\gamma^{-\frac{1}{2}} - \frac{1}{\alpha}D_a(\pi\omega^a)\gamma^{-\frac{1}{2}}; \tag{87}$$

and substituting this and the forms (71) and (72), of $\mathcal{H}_G$ and $\mathcal{C}_{Ga}$ in Eq. (86), we obtain

$$R^\alpha{}_\beta k^\beta n_\alpha\big|_\Sigma = D_a(D^a\alpha + \pi^a{}_b\omega^b\gamma^{-\frac{1}{2}} - \frac{1}{2}\pi\omega^a\gamma^{-\frac{1}{2}}) = D_a(D^a\alpha - K^a{}_b\omega^b). \tag{88}$$

Consequently, Eq. (40) holds with R and $G^\alpha{}_\beta n_\alpha$ given by Eqs. (70), (71), and (72), and with $\Delta g_{\alpha\beta}$ defined as a function of $(\gamma_{ab}, \alpha, \omega^a)$, independent of π^{ab}.

Finally, combining Eq. (40), Eq. (73) and Eq. (82) in the lagrangian derivation, as we obtain Eq. (54),

$$\delta\left(Q-\sum_i Q_i\right) = \int_\Sigma (\bar{T}\Delta dS + \bar{\mu}\Delta dM_B + v^a\Delta dC_a)$$
$$+\frac{1}{16\pi}\int_\Sigma \{\delta\pi^{ab}[D_a\omega_b + D_b\omega_a + 2\alpha(\pi_{ab} - \frac{1}{2}\gamma_{ab}\pi)\gamma^{-\frac{1}{2}}] - \alpha\delta\mathcal{H} - \omega^a\delta\mathcal{C}_a$$
$$+\alpha[\delta\gamma_{ab}(G^{ab} - 8\pi S^{ab}) + 16\pi\,\xi_\alpha\nabla_\beta T^{\alpha\beta}]\gamma^{\frac{1}{2}}\}d^3x. \tag{89}$$

Here the last two integrals in Eq. (54) are combined by using $2\delta[(G^\alpha{}_\beta - 8\pi T^\alpha{}_\beta)k^\beta n_\alpha] = \delta\alpha\mathcal{H} + \alpha\delta\mathcal{H} + \delta\omega^a\mathcal{C}_a + \omega^a\delta\mathcal{C}_a$.

When the field equations are satisfied, and π^{ab} is given by

$$\pi^{ab} = -\frac{1}{\alpha}(D^{(a}\omega^{b)} - \gamma^{ab}D_c\omega^c)\gamma^{1/2}, \tag{90}$$

we have

$$\delta Q = \int_\Sigma \left[\bar{T}\Delta dS + \bar{\mu}\Delta dM_B + v^a\Delta dC_a\right] + \sum_i \kappa_i\delta A_i \tag{91}$$

IV. APPLICATION TO THE INSPIRALING BINARY BLACK HOLE – NEUTRON STAR SYSTEM

A. Comparing configurations in quasi-stationary systems

Our study of a generalized first law was spurred by the fact that equilibria stationary in a rotating frame – spacetimes with helical Killing vectors – are used in several approaches to binary inspiral. In each of these cases, one approximates the inspiral phase of binary coalescence by an evolutionary path through a sequence of equilibria. The first law has a strikingly simple form when used to compare such dynamically related spacetimes: For isentropic fluids, dynamical evolution conserves the baryon mass, entropy, and vorticity of each fluid element, and we show that the first law becomes

$$\delta Q = \frac{1}{8\pi}\sum_i \kappa_i\delta A_i; \tag{92}$$

or

$$\delta Q = 0, \tag{93}$$

for perfect fluid spacetime with no black holes. In the gauge that we have chosen ($\delta k^\alpha = 0$), when the spacetime is asymptotically flat and k^α has the asymptotic form $t^\alpha + \Omega\phi^\alpha$, with t^α and ϕ^α timelike and rotational Killing vectors of a flat asymptotic metric, we find

$$\delta Q = \delta M - \Omega\delta J, \tag{94}$$

with M and J the mass and angular momentum at spatial infinity. In particular, the first law in this form describes (i) comoving binaries, flows with $v^\alpha = 0$; and (ii) irrotational binaries, potential flows $hu_\alpha = \nabla_\alpha\Phi$, with

$$\Delta(hu_\alpha) = \nabla_\alpha\Delta\Phi. \tag{95}$$

For an isentropic fluid, conservation of rest mass, entropy, and vorticity have the form

$$\pounds_u(\rho\sqrt{-g}) = 0, \quad \pounds_u s = 0, \quad \pounds_u\omega_{\alpha\beta} = 0, \tag{96}$$

with the relativistic vorticity $\omega_{\alpha\beta}$ given by

$$\omega_{\alpha\beta} = q_\alpha{}^\gamma q_\beta{}^\delta\left[\nabla_\gamma(hu_\delta) - \nabla_\delta(hu_\gamma)\right] = \nabla_\alpha(hu_\beta) - \nabla_\beta(hu_\alpha). \tag{97}$$

The perturbed conservation laws have the first integrals

$$\Delta(\rho u^\alpha\sqrt{-g}) = 0, \quad \Delta s = 0 \quad \Delta\omega_{\alpha\beta} = 0, \tag{98}$$

appropriate to the difference between two flows that are related by a dynamical evolution. It immediately follows that the first and second terms of Eq.(55) vanish for isentropic flows.

To see that the third term vanishes when the perturbed vorticity vanishes, we use $[d, \pounds_\xi] = 0$ to write

$$0 = \Delta\omega_{\alpha\beta} = \nabla_\alpha \Delta\left(hu_\beta\right) - \nabla_\beta \Delta\left(hu_\alpha\right), \tag{99}$$

implying $\Delta hu_\alpha = \nabla_\alpha \Delta\Phi$, as in Eq. (95). The third term in Eq. (55) can then be written

$$\int_\Sigma v^\beta \Delta(hu_\beta)\rho u^\alpha dS_\alpha = \int_\Sigma v^\beta \nabla_\beta \Delta\Phi \rho u^t k^\alpha dS_\alpha \tag{100}$$

$$= \int_\Sigma \nabla_\beta (v^\beta \Delta\Phi \rho u^t) k^\alpha dS_\alpha - \int_\Sigma \nabla_\beta (v^\beta \rho u^t)\Delta\Phi k^\alpha dS_\alpha \tag{101}$$

The first term in this last equality vanishes, because it is the integral of a total divergence. (Write $(\nabla_\beta A^\beta)k^\alpha = \nabla_\beta(A^\beta k^\alpha - A^\alpha k^\beta)$ and use Stokes' theorem; or, more concretely, write $k^\alpha dS_\alpha = \sqrt{-g}d^3x$.) For the second term, recalling the definition of v^α in Eq. (10), we have

$$\nabla_\beta(v^\beta \rho u^t) = \nabla_\beta(\rho u^\beta) - \pounds_k(\rho u^t) - \rho u^t \nabla_\beta k^\beta, \tag{102}$$

with each term on the right separately vanishing.

Thus, for spacetimes related by a perturbation that locally conserves baryon mass, entropy and vorticity, the first law has the form (92), as claimed.

B. Asymptotically flat systems

We will use the 3+1 formalism of Sec. (III A) to evaluate $\delta Q = \delta Q_K + \delta Q_L$. In the post-Newtonian and in the Isenberg-Wilson-Mathews spacetimes that have been used to describe binary systems, the 3-metric has the asymptotic form

$$\gamma_{ab} = f_{ab} + O(r^{-1}), \tag{103}$$

where $r = (\delta_{ij}x^i x^j)^{1/2}$, with $\{x^i\}$ a chart for which $f_{ij} = \delta_{ij}$.

By writing $k^\alpha = \alpha n^\alpha + \omega^\alpha$, as in Sec. (III A), we choose a shift ω^a associated with a comoving chart at spatial infinity. That is,

$$\omega^a = \Omega\phi^a + \beta^a, \quad \text{where} \quad \beta^a = O(r^{-2}), \tag{104}$$

and ϕ^a is a rotational Killing vector of the flat metric f_{ab}

$$\phi^a = x^1(\partial_2)^a - x^2(\partial_1)^a. \tag{105}$$

The extrinsic curvature and lapse have asymptotic behavior

$$K_{ab} = O(r^{-3}), \qquad \alpha = 1 + O(r^{-1}), \qquad D_a\alpha = O(r^{-2}). \tag{106}$$

To evaluate δQ, we first define two asymptotic masses and the asymptotic angular momentum. A mass M_K seen by a test particle in Keplerian orbit is associated with the asymptotic form of the lapse,

$$M_K := \frac{1}{4\pi}\int_\infty D^a\alpha\, dS_a = \lim_{r\to\infty}\frac{1}{4\pi}\int_{S_r} \partial_r\alpha\, r^2 d\Omega, \tag{107}$$

where

$$\int_\infty := \lim_{r\to\infty}\int_{S_r}$$

with S_r a sphere of constant r. In terms of the metric potentials, M_K has the form of the Komar mass associated with an timelike asymptotic Killing vector t^α.

The ADM mass is computed from the 3-metric:

$$M_{\rm ADM} = \frac{1}{16\pi}\int_\infty (f^{ac}f^{bd} - f^{ab}f^{cd})\partial_b\gamma_{cd}\, dS_a$$
$$= -\lim_{r\to\infty}\frac{1}{2\pi}\int_{S_r}\partial_r\psi\ r^2 d\Omega. \tag{108}$$

Finally, the angular momentum associated with the asymptotic rotational Killing vector is given by

$$J = -\frac{1}{8\pi}\int_\infty \pi^a{}_b\ \phi^b dS_a = \frac{1}{8\pi}\int_\infty K^a{}_b\ \phi^b dS_a. \tag{109}$$

As in the first equality of Eq. (84), we have

$$Q_K = -\frac{1}{8\pi}\int_\infty (-D^a\alpha + K^a{}_b\omega^b)dS_a = \frac{1}{8\pi}\int_\infty (D^a\alpha - K^a{}_b\Omega\phi^b)dS_a, \tag{110}$$

whence, by Eqs. (107) and (109)

$$Q_K = \frac{1}{2}M_K - \Omega J. \tag{111}$$

We obtain δQ_L from Eqs. (82) and (76). Using $\phi^a D_a r = 0$ and the asymptotic behavior given above, we have

$$\tilde\Theta^a = \frac{1}{16\pi}\{[-2f^{1/2}\delta D^a\alpha - 2\delta(\pi^a{}_b\,\Omega\phi^b) + 2\delta\pi^a{}_b\,\Omega\phi^b]f^{-1/2}$$
$$+ (f^{ac}f^{bd} - f^{ab}f^{cd})D_b\delta\gamma_{cd}\} + O(r^{-3}), \tag{112}$$

$$\delta\tilde Q_L = \frac{1}{16\pi}[-2\delta\int_\infty D^a\alpha\ dS_a + 2\delta\int_\infty K^a{}_b\Omega\phi^b dS_a - 2\Omega\ \delta\int_\infty K^a{}_b\phi^b dS_a$$
$$+ \int_\infty (f^{ac}f^{bd} - f^{ab}f^{cd})D_b\delta\gamma_{cd}dS_a\,]$$
$$= \quad -\frac{1}{2}\delta M_K + \delta(\Omega J) - \Omega\delta J + \delta M_{\rm ADM}$$
$$= \quad \delta M_{\rm ADM} - \frac{1}{2}\delta M_K + \delta\Omega J. \tag{113}$$

Adding Eq. (113) to $\delta(111)$, we have

$$\delta Q = \frac{1}{2}\delta M_K - \delta(\Omega J) + \delta M_{\rm ADM} - \frac{1}{2}\delta M_K + \delta\Omega J$$
$$= \delta M_{\rm ADM} - \Omega\delta J, \tag{114}$$

in agreement with the usual first law.

C. The first law for spacetimes with a conformally flat spatial geometry

As mentioned earlier, several groups have recently obtained quasi-equilibrium sequences [6–9], approximating binary inspiral by a sequence of Isenberg-Wilson-Mathews spacetimes (henceforth *IWM spacetimes*), spacetimes whose 3-geometry is conformally flat and whose five metric potentials satisfy a truncated set of five field equations. More precisely, the metric of a IWM spacetime satisfies the constraint equations and the spatial trace of the Einstein equation, together with the maximal slicing condition for its conformally flat slices; and its matter satisfies the equation of motion, $\nabla_\beta T^{\alpha\beta} = 0$ (see e.g., Isenberg [35] or Flanagan [36]).

As Detweiler has pointed out, when the spacetime has a helical (or timelike) Killing vector, one cannot in general solve all of these equations simultaneously for a metric with conformally flat spacelike slices. One must omit one relation to accommodate the new constraint that the existence of a Killing vector imposes on the extrinsic curvature,

K_{ab}. We note first that, if one omits the $K = 0$ condition, the resulting spatially conformally flat spacetime satisfies an exact first law, despite the fact that only a truncated set of field equations are imposed.

In the second part of this section, we note that one can alternatively retain the $K = 0$ condition if one simply *defines* a tensor $\widehat{K}_{ab}$ by the form (Eq.(123) below) that the extrinsic curvature would take in a spacetime with a helical Killing vector foliated by $K = 0$ slices. We show that the first law is *exact* in this framework. This is surprising, in view of the artificiality of the definition of $\widehat{K}_{ab}$ and the fact that one component of the Einstein equation is not satisfied in the IWM framework.

In each case, one has a spacetime foliated by hypersurfaces whose spatial metrics have the form

$$\gamma_{ab} = \psi^4 f_{ab}, \tag{115}$$

with f_{ab} a flat metric. The corresponding 4-tensors,

$$\gamma_{\alpha\beta} = \psi^4 f_{\alpha\beta}, \tag{116}$$

are Lie derived by the Killing vector k^α:

$$\pounds_k \gamma_{\alpha\beta} = 0, \quad \pounds_k \psi = 0, \quad \pounds_k f_{\alpha\beta} = 0. \tag{117}$$

In particular (although we will not use the fact in this section),

$$k^\alpha = t^\alpha + \Omega\phi^\alpha, \tag{118}$$

with ϕ^a a rotational Killing vector of f_{ab}.

In the first case (with K not required to vanish), the spacetime satisfies on each Σ_t the equations

$$\mathcal{H} = 0, \;\; \mathcal{C}_a = 0, \;\; \gamma_{ab}(G^{ab} - 8\pi T^{ab}) = 0, \;\; \nabla_\beta T^{\alpha\beta} = 0, \tag{119}$$

together with the relation (90) expressing π^{ab} in terms of the metric. Because

$$\delta\gamma_{ab} = 4\frac{\delta\psi}{\psi}\gamma_{ab}, \tag{120}$$

it is exactly this set of equations that occur in the action and in the 3+1 form of the first law (89), when one compares two spatially conformally flat spacetimes.

Finally, comparing asymptotically flat spacetimes of this kind, with no local change in entropy, baryon number, or circulation, we have

$$\delta M = \Omega\delta J + \sum \kappa_i \delta A_i. \tag{121}$$

We consider next solutions $(\hat{\pi}^{ab}, \gamma_{ab}, \alpha, \omega^a, \epsilon, u^\alpha)$, to the same set (119) of equations, now with $\hat{\pi} = 0$:

$$\hat{\pi}^{ab} = -\hat{K}^{ab}\gamma^{\frac{1}{2}}, \tag{122}$$

with $\hat{K}_{ab}$ the *tracefree part* of the extrinsic curvature:

$$\hat{K}_{ab} = \frac{1}{2\alpha}(D_a\omega_b + D_b\omega_a - \frac{2}{3}\gamma_{ab}D_c\omega^c). \tag{123}$$

One writes $\mathcal{H}$, $\mathcal{C}_a$, $\gamma_{ab}(G^{ab} - 8\pi T^{ab})$, and $\nabla_\beta T^{\alpha\beta}$ as they occur in the Hamiltonian formalism (for the metric), as functions of $(\pi^{ab}, \gamma_{ab}, \alpha, \omega^a)$ and the matter variables; one substitutes for γ_{ab} and π^{ab} the expressions

$$\gamma_{ab} = \psi^4 f_{ab}, \quad \hat{\pi}^{ab} = -\frac{1}{2\alpha}(D^a\omega^b + D^b\omega^a - \frac{2}{3}\gamma^{ab}D_c\omega^c)\gamma^{-\frac{1}{2}}; \tag{124}$$

and one solves the resulting system of equations for $(\psi, \alpha, \omega^a; \epsilon, v^a)$.

$$\hat{\mathcal{H}} = 0, \quad \hat{\mathcal{C}}_a = 0, \quad (\hat{G}^{ab} - 8\pi S^{ab})\gamma_{ab} = 0, \quad \nabla_\beta T^{\alpha\beta} = 0, \tag{125}$$

where

$$\hat{\mathcal{H}} = \mathcal{H}(\hat{\pi}^{ab}, \gamma_{ab}, \alpha, \omega^a; \epsilon, u^\alpha), \quad \hat{\mathcal{C}}^a = \mathcal{C}^a(\hat{\pi}^{ab}, \gamma_{ab}, \alpha, \omega^a; \epsilon, u^\alpha), \quad \hat{G}^{ab}\gamma_{ab} = G^{ab}\gamma_{ab}(\hat{\pi}^{ab}, \gamma_{ab}, \alpha, \omega^a; \epsilon, u^\alpha). \tag{126}$$

Then, for a family of such solutions, the quantities $\delta\alpha$, $\delta\omega$, and $\delta\gamma_{ab} = 4\frac{\delta\psi}{\psi}\gamma_{ab}$ occurring on the right of the first law (89) multiply expressions that vanish. Because $\delta\hat{\pi}^{ab}$ is traceless, the expression involving $\delta\hat{\pi}^{ab}$ has the form

$$\frac{1}{16\pi}\delta\hat{\pi}^{ab}[D_a\omega_b + D_b\omega_a + 2\alpha\gamma^{-\frac{1}{2}}\hat{\pi}_{ab}] = \frac{1}{16\pi}\delta\hat{\pi}^{ab}[D_a\omega_b + D_b\omega_a - \frac{2}{3}\gamma_{ab}D_c\omega^c + 2\alpha\gamma^{-\frac{1}{2}}\hat{\pi}_{ab}] = 0. \tag{127}$$

Eq. (89) thus yields

$$\delta\hat{Q}_L - \frac{1}{8\pi}\delta\int \hat{R}^\alpha{}_\beta k^\beta dS_\alpha = \int_\Sigma [\overline{T}\Delta dS + \overline{\mu}\Delta dM_B + v^a\Delta dC_a]. \tag{128}$$

To recover the first law in the form

$$\delta M = \Omega\delta J, \tag{129}$$

we must show that

$$-\frac{1}{8\pi}\delta\int_\Sigma \hat{R}^\alpha{}_\beta k^\beta dS_\alpha = -\frac{1}{8\pi}\delta\int_{\partial\Sigma} \nabla^\alpha k^\beta dS_{\alpha\beta}. \tag{130}$$

This is not obvious, because, in replacing the extrinsic curvature by its tracefree part, we invalidate the Killing identity (35):

$$\nabla_\beta\nabla^\alpha k^\beta \neq R^\alpha{}_\beta(\hat{\pi}^{ab}, \psi, \alpha, \omega^a)k^\beta \tag{131}$$

Remarkably, however, the n_α components of the two sides of this inequality differ by a divergence; and the asymptotic behavior of the spacetime implies the equality

$$Q_K = -\frac{1}{8\pi}\int_\infty \nabla^\alpha k^\beta dS_{\alpha\beta} = -\frac{1}{8\pi}\int \hat{R}^\alpha{}_\beta k^\beta dS_\alpha \tag{132}$$

That is, from Eq. (85), we have

$$\nabla_\beta\nabla^\alpha k^\beta n_\alpha\big|_\Sigma = R^\alpha{}_\beta k^\beta n_\alpha|_\Sigma = D_a(D^a\alpha - K^a{}_b\omega^b); \tag{133}$$

and

$$\hat{R}^\alpha{}_\beta k^\beta n_\alpha\Big|_\Sigma = D_a(D^a\alpha - \hat{K}^a{}_b\omega^b). \tag{134}$$

Then

$$\begin{aligned}\nabla_\beta\nabla^\alpha k^\beta n_\alpha\big|_\Sigma - \hat{R}^\alpha{}_\beta k^\beta n_\alpha|_\Sigma &= D_a[(\hat{K}^a{}_b - K^a{}_b)\omega^b] \\ &= -\frac{1}{3}D_a(\omega^a K).\end{aligned} \tag{135}$$

As noted in Sect. (IV B),

$$\omega^a = \Omega\phi^a + \beta^a, \quad \text{with} \quad \beta^a = O(r^{-2}), \tag{136}$$

where ϕ^a is a rotational Killing vector of the flat metric f_{ab}; and $K = O(r^{-3})$. [7] We then have

[7] If, however, one allows a nonzero 3-momentum, with boosted-Schwarzschild asymptotics, then $\psi = 1 + f(\hat{r})/r + O(r^{-2})$ and and $\beta^a = O(r^{-1})$. Because K is given by $\frac{1}{\alpha}(\frac{6}{\psi}\Omega\phi^a D_a\psi + D_a\beta^a)$, one can only demand $K = O(r^{-1})$, $\beta^a K = O(r^{-2})$, allowing a finite contribution to Q_K.

$$
\begin{aligned}
Q_K = -\frac{1}{8\pi}\int R^\alpha{}_\beta k^\beta dS_\alpha &= -\frac{1}{8\pi}\int_\Sigma \hat{R}^\alpha{}_\beta k^\beta dS_\alpha - \frac{1}{24\pi}\int_{\partial\Sigma}\omega^b K dS_b \\
&= -\frac{1}{8\pi}\int_\Sigma \hat{R}^\alpha{}_\beta k^\beta dS_\alpha,
\end{aligned}
\tag{137}
$$

as claimed.

From Eq. (108), we conclude

$$
\delta M = \Omega\delta J, \tag{138}
$$

along a family of conformally flat solutions to the IWM equations, written in terms of $(\hat{\pi}^{ab}, \psi, \alpha, \omega^a)$.

Note that the equation $\xi_\beta \nabla_\alpha T^{\alpha\beta} = 0$ is satisfied, because, for an isentropic fluid, the equation of hydrostatic equilibrium, conservation of rest-mass, and the one-parameter equation of state together imply $\nabla_\alpha T^{\alpha\beta} = 0$. To see this explicitly, we decompose the divergence of the stress tensor as follows:

$$
\nabla_\alpha T^{\alpha\beta} = (q_{\beta\gamma} - u_\beta u_\gamma)\nabla_\alpha T^{\alpha\gamma} = \rho\,[u^\gamma\nabla_\gamma(hu_\beta) + \nabla_\beta h] + u_\beta h\nabla_\alpha(\rho u^\alpha) - \rho T\nabla_\beta\, s \tag{139}
$$

In constructing an isentropic ($s = \text{const}$) equilibrium model, conservation of rest mass is assumed, and a barotropic equation of state $p = p(\rho)$ is used. Helical symmetry and the assumption that the fluid flow is either co-rotational or irrotational then leads to a first integral of the Euler equation $u^\gamma\nabla_\gamma(hu_\beta) + \nabla_\beta h = 0$. It is this first integral, specialized to a conformally flat metric, that is solved in the IWM formalism, implying that $\nabla_\alpha T^{\alpha\beta} = 0$. Thus, as claimed, all terms involving the field equations vanish in Eq. (54), and the first law holds for IWM spacetimes in the form (55).

As in the exact theory, when the system includes black holes, the $\kappa_i\delta A_i$ terms refer to Killing horizons. The IWM spacetimes do not satisfy the Raychaudhuri equation for the null generators of the horizon; as a result, as noted in the introduction, Killing horizons in IWM spacetimes need not co-rotate with the orbital motion.

V. DISCUSSION

The first law can be used to deduce a criterion for orbital stability for the asymptotically flat models of binary equilibria discussed above, using a theorem of Sorkin [37]. Consider a one-parameter family $\mathcal{Q}(\lambda)$ of binary equilibrium models along which baryon number, entropy and circulation are locally constant (the lagrangian changes $\Delta s, \Delta dM_B$, and ΔdC_α vanish). Suppose that $\dot{J} = 0$ at a point λ_0 along the sequence, and that $\dot{\Omega}\dot{J} \neq 0$ at λ_0. Then the part of the sequence for which $\dot{\Omega}\dot{J} > 0$ is unstable for λ near λ_0.

The result relies on a first law in the form

$$
dM = \Omega dJ \tag{140}
$$

and on the fact that the equilibria are extrema of mass with J constant. As we have seen, this is the case for a configuration space in which baryon number, entropy, and circulation are fixed for each fluid element. For asymptotically flat models with one or more black holes, if one also fixes the area of the horizon along a sequence, then the same criterion above can be used to diagnose stability.

In general, the proof of the theorem shows only that the spacetime is *secularly* unstable on one side of the turning point. In the present context, however, the theorem shows the existence of nearby configurations with lower mass that can be reached by perturbations that conserve baryon number, entropy, and circulation; this suggests that the criteria locates the onset of dynamical instability.

When one models stationary binary systems in full GR, the lack of asymptotic flatness leads to several ambiguities. For binary charges in Minkowski space, one can obtain a one-parameter family of equilibria if one simply replaces asymptotic regularity (finite energy) by a condition that the electromagnetic field be given by the half-advanced + half-retarded Green's function. In GR, it remains to be seen whether one can find an analogous asymptotic condition. Simply requiring equal amounts of ingoing and outgoing radiation is a weaker condition even in Minkowski space; in GR one must have asymptotic conditions as restrictive as asymptotic flatness to avoid ambiguity in each asymptotic multipole. Finally, as mentioned in Sect. III, the helical Killing vector has an arbitrary scaling that one must resolve to obtain a unique value for the charge Q of the first law.

ACKNOWLEDGMENTS

We thank Ric Ancel, Abhay Ashtekar, Steven Detweiler, Eric Gourgoulhon, James Isenberg, Stephen Hawking and Robert Wald for helpful conversations; and we also thank Detweiler and Gourgoulhon for reading and pointing out errors in an earlier draft.

This research was supported in part by NSF Grant PHY00-71044 and by a Japanese Monbu-kagakusho Grant (No. 13740143).

APPENDIX A: VIRIAL RELATION IN IWM SPACETIMES

In this appendix, we derive a virial relation for quasiequilibrium states in IWM spacetimes. Incidentally, we show that the virial relation is equivalent to the relation $M_K = M_{\rm ADM}$. [8]

As described in IV C, we use a 3+1 formalism, with 3-metric $\gamma_{ab} = \psi^4 f_{ab}$, and with a helical Killing vector that has the form

$$k^\alpha = t^\alpha + \Omega\phi^\alpha, \tag{A1}$$

where ϕ^a is a rotational Killing vector of the flat metric f_{ab}. Throughout this appendix, we use Cartesian coordinates $t, \{x^i\}$ for which f_{ab} has components $f_{ij} = \delta_{ij}$ and $t^\mu = \delta^\mu_0$. In the IWM formalism, one imposes the maximal slicing condition $K = 0$ on the family of $t =$const surfaces Σ_t; and, instead of solving the full Einstein equation, one solves the Hamiltonian constraint, the momentum constraint and the equation for the slicing condition. Here, however, as in Sect. IV C, to obtain a set of equations consistent with the existence of a helical Killing vector, we replace the extrinsic curvature in this set of equations by its tracefree part, $\hat{K}^{ij}$.

The basic equations are then

$$\Delta\psi = -2\pi\psi^5\rho_H - \frac{\psi^5}{8}\hat{K}_i{}^j\hat{K}_j{}^i \equiv -S_\psi, \tag{A2}$$

$$\partial_j(\sqrt{\gamma}\hat{K}_i{}^j) = 8\pi j_i\sqrt{\gamma}, \tag{A3}$$

$$\Delta\chi = 2\pi\chi\psi^4(\rho_H + 2S_k{}^k) + \frac{7}{8}\chi\psi^4\hat{K}_i{}^j\hat{K}_j{}^i \equiv S_\chi, \tag{A4}$$

where Δ denotes the flat Laplacian for three space, $\sqrt{\gamma} = \det(\gamma_{ij}) = \psi^6$, and $\chi \equiv \alpha\psi$. (See Eqs. (74) for definition of ρ_H, j_i and S_{ij}.) The energy-momentum tensor is assumed to be nonzero only inside the light cylinder $(x^2+y^2)^{1/2} < \Omega^{-1}$.

The shift vector β^a of Eq. (104) satisfies

$$\beta^\alpha = -\alpha n^\alpha + t^\alpha. \tag{A5}$$

The Cartesian components $\hat{K}_i^j$ are given in terms of β^i by

$$\hat{K}_i{}^j = \frac{1}{2\alpha}\left(\partial_i\beta^j + \delta_{il}\delta^{jk}\partial_k\beta^l - \frac{2}{3}\delta_i^j\partial_k\beta^k\right). \tag{A6}$$

The asymptotic behavior of geometric variables is that of Eqs. (103)- (106),

$$\psi = 1 + O(r^{-1}), \tag{A7}$$

$$\chi = 1 + O(r^{-1}), \tag{A8}$$

$$\beta^i = O(r^{-2}), \tag{A9}$$

$$\hat{K}_i{}^j = O(r^{-3}), \tag{A10}$$

appropriate for an asymptotically flat spacetime in a chart for which the total ADM 3-momentum vanishes:

---

[8] The relation $M_K = M_{\rm ADM}$ for stationary and asymptotically flat spacetimes has been proven by Beig [38], and by Ashtekar and Magnon-Ashtekar [39]. A virial relation relying on this has been derived by Gourgoulhon and Bonazzola [40].

$$P_i \equiv \frac{1}{8\pi}\oint_{r\to\infty} K_i^{\ j}\sqrt{\gamma}d\sigma_j = 0, \tag{A11}$$

where $d\sigma_j = (D_j r)r^2 d\Omega$ and $\sqrt{\gamma}$ is computed in Cartesian coordinates.

We now derive the virial relation and show the equivalence of Komar and ADM mass for quasiequilibria of two neutron stars. For $r\to\infty$, χ and ψ behave as $1-M_\chi/2r+O(r^{-2})$ and $1+M_{\rm ADM}/2r+O(r^{-2})$. From this asymptotic behavior, we can define M_χ and $M_{\rm ADM}$ by the surface integrals,

$$\begin{aligned} M_\chi &= \frac{1}{2\pi}\oint_{r\to\infty}\delta^{ij}\psi\partial_i\chi d\sigma_j \\ M_{\rm ADM} &= -\frac{1}{2\pi}\oint_{r\to\infty}\delta^{ij}\chi\partial_i\psi d\sigma_j. \end{aligned} \tag{A12}$$

Since $M_\chi = -M_{\rm ADM} + 2M_K$, our goal is to show $M_\chi = M_{\rm ADM}$.

Using Gauss's law, they can be rewritten in the manner

$$M_\chi = \frac{1}{2\pi}\int(\psi S_\chi + \delta^{ij}\partial_i\chi\partial_j\psi)d^3x, \tag{A13}$$

$$M_{\rm ADM} = \frac{1}{2\pi}\int(\chi S_\psi - \delta^{ij}\partial_i\psi\partial_j\chi)d^3x. \tag{A14}$$

We next derive a relation that will be used several times in the calculations that follow. From $\chi\psi^5\hat{K}_i^{\ j}\hat{K}_j^{\ i} = \sqrt{\gamma}\hat{K}_i^{\ j}\partial_j\beta^i$, we have

$$\begin{aligned} &\int \chi\psi^5\hat{K}_i^{\ j}\hat{K}_j^{\ i}d^3x = \int\sqrt{\gamma}\hat{K}_i^{\ j}\partial_j\beta^i d^3x \\ &= -\int\partial_j(\sqrt{\gamma}\hat{K}_i^{\ j})\beta^i d^3x + \oint\sqrt{\gamma}\hat{K}_i^{\ j}\beta^i d\sigma_j \\ &= -8\pi\int\sqrt{\gamma}j_i\beta^i d^3x, \end{aligned} \tag{A15}$$

where we use the asymptotic behaviors at $r\to\infty$ and Eq. (A3) to obtain the last line. $\oint$ without specification of a surface denotes a surface integral over $\partial\Sigma$: $\oint = \oint_{r\to\infty}$. From the vanishing of the total ADM 3-momentum (more precisely, from the vanishing of $\int_{r\to\infty}\hat{K}_i^j\sqrt{\gamma}dS_j$) and from the momentum constraint (A3), we have

$$0 = \int j_i\sqrt{\gamma}d^3x, \tag{A16}$$

which may be interpreted as the linear momentum of a neutron star.

Using Eqs. (A13) and (A14), we write the difference between $M_{\rm ADM}$ and M_χ in the form

$$\begin{aligned} M_\chi - M_{\rm ADM} &= \frac{1}{\pi}\int\left[2\pi\chi\psi^5 S_k^{\ k} + \frac{3}{8}\chi\psi^5 K_i^{\ j}K_j^{\ i} + \delta^{ij}\partial_i\psi\partial_j\chi\right]d^3x \\ &= 2\int\left[\sqrt{\gamma}\{j_k v^k + 3\alpha P\} - \frac{1}{2}\sqrt{\gamma}j_j\beta^j + \frac{1}{2\pi}\delta^{ij}\partial_i\psi\partial_j\chi\right]d^3x, \end{aligned} \tag{A17}$$

where we use $S_k^{\ k} = j_k u_l\tilde{\gamma}^{kl}/(\alpha u^t) + 3P$, $v^k = u^k/u^t$[9], $\gamma^{kl}u_l = u^t(v^k+\beta^k)$ and Eq. (A15). In the following we show that the relation $M_\chi = M_{\rm ADM}$ is equivalent to the virial relation.

To derive the virial relation, we first write the general relativistic Euler equation $\gamma^\nu_{\ k}\nabla_\mu T^\mu_{\ \nu} = 0$ in the form

$$\partial_t(j_k\sqrt{\gamma}) + \partial_k(j_l\sqrt{\gamma}v^l) + \partial_k(\alpha\sqrt{\gamma}P) + \rho_H\psi^5\partial_k\chi - (\rho_H + 2S_l^{\ l})\chi\psi^4\partial_k\psi - \sqrt{\gamma}j_l\partial_k\beta^l + \frac{1}{2}\chi\psi S_{ij}\partial_k\tilde{\gamma}^{ij} = 0, \tag{A18}$$

[9]The above definition for v^k is used only in this appendix. Note that v^k was differently used for spatial velocity vector in co-moving frame with k^α as defined in Eq.(10) in main sections.

where $\tilde{\gamma}^{ij} = \gamma^{ij}\psi^4$. Equation (A18) is a fully general relativistic expression. In the IWM spacetimes, $\tilde{\gamma}^{ij} = \tilde{\gamma}_{ij} = \delta_{ij}$ and consequently, the last term in Eq. (A18) is neglected.

In the following calculation, we choose the x^1-axis so that, on some time-slice Σ_t, it lies along the centers of the two members of the binary system.

As in the Newtonian case, the virial relation can be derived by taking inner product with $\hat{x}^k$ and by performing an integral over three space, i.e.,

$$\int d^3\hat{x}\hat{x}^k \left[\partial_t(j_k\sqrt{\gamma}) + \partial_j(j_k\sqrt{\gamma}v^j) + \partial_k(\alpha\sqrt{\gamma}P) + \rho_H\psi^5\partial_k\chi - (\rho_H + 2S_l{}^l)\chi\psi^4\partial_k\psi - \sqrt{\gamma}j_l\partial_k\beta^l\right] = 0. \tag{A19}$$

Below, we shall carry out integrals separately. For simplicity, we omit hats ($\hat{}$) on indices in the following.

(1) First term: Since we assume the existence of the helical Killing vector, we have a relation

$$\partial_t\mathcal{J}_k = -\Omega[\partial_l(\phi^l\mathcal{J}_k) + \mathcal{J}_l\partial_k\phi^l], \tag{A20}$$

where we use $\partial_l\phi^l = 0$ and $\mathcal{J}_k \equiv j_k\sqrt{\gamma}$. In the present coordinates, $\phi^l = (-x^2, x^1 - b, 0)$. After an integration by parts, we obtain

$$\int x^k\partial_t\mathcal{J}_k d^3x = \Omega\int(\phi^k\mathcal{J}_k - \mathcal{J}_l x^k\partial_k\phi^l)d^3x. \tag{A21}$$

From a relation $x^k\partial_k\phi^l = \phi^l + \delta^{2l}b$, we immediately find

$$\int x^k\partial_t(j_k\sqrt{\gamma})d^3x = -b\Omega\int j_2\sqrt{\gamma}d^3x \equiv -b\Omega P_{\rm NS}, \tag{A22}$$

where $P_{\rm NS}$ is interpreted as the linear momentum of a neutron star [see Eq. (A16)].

(2) Second and third terms: An integration by parts immediately yields

$$\int x^k\partial_j(j_k v^j\sqrt{\gamma})d^3x = -\int j_k v^k\sqrt{\gamma}d^3x, \tag{A23}$$

$$\int x^k\partial_k(\alpha\sqrt{\gamma}P)d^3x = -3\int\alpha\sqrt{\gamma}Pd^3x. \tag{A24}$$

(3) Fourth and fifth terms: Using Eqs. (A2) and (A4), we can rewrite these terms as

$$\rho_H\psi^5\partial_k\chi - (\rho_H + 2S_l{}^l)\chi\psi^4\partial_k\psi = -\frac{1}{2\pi}\left[\Delta\psi\partial_k\chi + \Delta\chi\partial_k\psi\right] - \frac{\psi^{12}K_i{}^jK_j{}^i}{16\pi}\partial_k\left(\frac{\alpha}{\sqrt{\gamma}}\right). \tag{A25}$$

Taking into account an identity,

$$\int[(x^k\partial_k\psi)\Delta\chi + (x^k\partial_k\chi)\Delta\psi]d^3x = \int\delta^{ij}\partial_i\chi\partial_j\psi d^3x, \tag{A26}$$

we find

$$\begin{aligned}&\int x^k[\rho_H\psi^5\partial_k\chi - (\rho_H + 2S_l{}^l)\chi\psi^4\partial_k\psi]d^3x\\&= -\int\left[\frac{1}{2\pi}\delta^{ij}\partial_i\chi\partial_j\psi + \frac{\psi^{12}K_i{}^jK_j{}^i}{16\pi}x^k\partial_k\left(\frac{\alpha}{\sqrt{\gamma}}\right)\right]d^3x.\end{aligned} \tag{A27}$$

(4) Sixth term:

$$\begin{aligned}&-\int\sqrt{\gamma}j_ix^k\partial_k\beta^id^3x = -\frac{1}{8\pi}\int\partial_j(\sqrt{\gamma}K_i{}^j)x^k\partial_k\beta^id^3x\\&= \frac{1}{8\pi}\int[\sqrt{\gamma}K_i{}^jx^k\partial_k\partial_j\beta^i + \sqrt{\gamma}K_i{}^j\partial_j\beta^i]d^3x - \frac{1}{8\pi}\oint\sqrt{\gamma}K_i{}^jx^k\partial_k\beta^id\sigma_j\\&= -\frac{1}{8\pi}\int[\partial_k(\sqrt{\gamma}K_i{}^j)x^k\partial_j\beta^i + 2\sqrt{\gamma}K_i{}^j\partial_j\beta^i]d^3x + \frac{1}{8\pi}\oint\sqrt{\gamma}(K_i{}^jx^k - K_i{}^kx^j)\partial_j\beta^id\sigma_k\end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{8\pi}\int\left[\frac{\alpha x^k}{2\sqrt{\gamma}}\partial_k(\psi^{12}K_i{}^jK_j{}^i) + 2\alpha\sqrt{\gamma}K_i{}^jK_j{}^i\right]d^3x + \frac{1}{8\pi}\oint\sqrt{\gamma}(K_i{}^jx^k - K_i{}^kx^j)\partial_j\beta^i d\sigma_k \\
&= -\frac{1}{16\pi}\int\left[\alpha\sqrt{\gamma}K_i{}^jK_j{}^i - \psi^{12}K_i{}^jK_j{}^ix^k\partial_k\left(\frac{\alpha}{\sqrt{\gamma}}\right)\right]d^3x \\
&\quad - \frac{1}{16\pi}\oint\alpha\psi^6K_i{}^jK_j{}^ix^kd\sigma_k + \frac{1}{8\pi}\oint\sqrt{\gamma}(K_i{}^jx^k - K_i{}^kx^j)\partial_j\beta^i d\sigma_k \\
&= \frac{1}{16\pi}\int\left[8\pi\sqrt{\gamma}j_k\beta^k + \psi^{12}K_i{}^jK_j{}^ix^k\partial_k\left(\frac{\alpha}{\sqrt{\gamma}}\right)\right]d^3x \\
&\quad + \frac{1}{16\pi}\oint\alpha\psi^6K_i{}^jK_j{}^ix^kd\sigma_k - \frac{1}{8\pi}\oint\sqrt{\gamma}K_i{}^kx^j\partial_j\beta^i d\sigma_k,
\end{aligned} \tag{A28}
$$

where we use Eqs. (A3) and (A15). Because of the asymptotic behavior, the surface terms at $r \to \infty$ in the last line of Eq. (A28) vanish. Eq. (A16) implies that the center of mass of the system does not move in the x^2 direction, that the sum of the momenta of the neutron stars vanishes.

Gathering the results of (1)–(6), we obtain the relation

$$
0 = -\int\left(j_kv^k\sqrt{\gamma} + 3\alpha\sqrt{\gamma}P + \frac{1}{2\pi}\delta^{ij}\partial_i\chi\partial_j\psi - \frac{1}{2}\sqrt{\gamma}j_k\beta^k\right)d^3x. \tag{A29}
$$

This is the virial relation for a neutron star binary system in quasiequilibrium.

From Eq. (A17), the right-hand side of Eq. (A29) is written as

$$
0 = -\frac{M_\chi - M_{\rm ADM}}{2}, \tag{A30}
$$

implying $M_{\rm ADM} = M_\chi = M_K$, if the virial relation holds.

APPENDIX B: THE FIRST LAW FOR NEWTONIAN BINARY SYSTEMS

In this appendix, we derive a first law of thermodynamics for Newtonian gravity. We start with a first-order perturbation of the energy of a perfect-fluid.

$$
E = T + W + U, \tag{B1}
$$

where

$$
T = \int_V \frac{1}{2}\rho\, v^2 dV, \qquad W = \int_V\left(\rho\,\Phi_{\rm N} + \frac{1}{8\pi G}\nabla_i\Phi_{\rm N}\nabla_i\Phi_{\rm N}\right)dV, \qquad U = \int_V \rho\, u\, dV. \tag{B2}
$$

and $\Phi_{\rm N}$ and u denote the Newtonian potential and specific internal energy[10]. An integral equation

$$
\delta\int_V f\,\rho\, dV = \int_V \Delta f\,\rho\, dV + \int_V f\,\Delta(\rho\, dV), \tag{B3}
$$

is satisfied for a perturbation.

The perturbation of the kinetic energy T can be expressed as follows

$$
\delta T = \int_V \rho v^i\Delta v_i + \int_V \frac{1}{2}v^2\Delta(\rho\, dV) + \int_V\left[\xi^j v_j\nabla_i(\rho v^i) + \rho\,\xi^j v^i\nabla_i v_j\right]dV - \oint_{\partial V}\rho\, v^i v_j\xi^j dS_i \tag{B4}
$$

where we used the relation,

[10] We use v^i for the fluid velocity vector in the inertial frame in this appendix. Note that it was differently used for spatial velocity vector in co-moving frame with k^α as defined in Eq.(10) in main sections.

$$\frac{1}{2}\Delta v^2 = v^i \Delta v_i - v^i v_j \nabla_i \xi^j. \tag{B5}$$

The perturbation of the gravitational potential energy becomes

$$\delta W = \int_V \Phi_{\rm N} \Delta(\rho\, dV) + \int_V \rho\, \xi^i \nabla_i \Phi_{\rm N}\, dV - \frac{1}{4\pi G}\int_V (\nabla^2 \Phi_{\rm N} - 4\pi G\rho)\, \delta\Phi_{\rm N}\, dV + \frac{1}{4\pi G}\oint_{\partial V} \nabla_i \Phi_{\rm N}\, \delta\Phi_{\rm N}\, dS_i. \tag{B6}$$

The perturbation of the internal energy becomes

$$\delta U = \int_V \rho\, T \Delta s\, dV + \int_V \left(u + \frac{P}{\rho}\right)\Delta(\rho\, dV) + \int_V \xi^i \nabla_i P\, dV - \oint_{\partial V} \xi^i\, P\, dS_i, \tag{B7}$$

where we used a relation

$$\nabla_i \xi^i dV = \Delta(dV), \tag{B8}$$

as well as a local thermodynamic relation,

$$\Delta u = T\Delta s + \frac{P}{\rho^2}\Delta\rho. \tag{B9}$$

Surface integrals appeared in expressions for δT, δW and δU are all vanish. Combining Eqs.(B4), (B6) and (B7), we have a perturbation of the Newtonian energy integral:

$$\begin{aligned}
\delta E &= \delta T + \delta W + \delta U \\
&= \int_V \rho\, T\Delta s\, dV + \int_V \left(\frac{1}{2}\, v^2 + \Phi_{\rm N} + u + \frac{P}{\rho}\right)\Delta(\rho\, dV) + \int_V \rho v^i \Delta v_i\, dV \\
&\quad + \int_V \left[\xi^j v_j \nabla_i(\rho v^i) + \xi^i \rho\left(v^j \nabla_j v_i + \frac{1}{\rho}\nabla_i P + \nabla_i \Phi_{\rm N}\right) - \frac{1}{4\pi G}(\nabla^2\Phi_{\rm N} - 4\pi G\rho)\,\delta\Phi_{\rm N}\right] dV \\
&= \int_V \rho\, T\Delta s\, dV + \int_V \left(\frac{1}{2}\, v^2 + \Phi_{\rm N} + u + \frac{P}{\rho}\right)\Delta(\rho\, dV) + \int_V \rho v^i \Delta v_i\, dV \\
&\quad + \int_V \xi^j v_j \left[\frac{\partial \rho}{\partial t} + \nabla_i(\rho v^i)\right] dV + \int_V \xi^i \rho\left(\frac{\partial v_i}{\partial t} + v^j \nabla_j v_i + \frac{1}{\rho}\nabla_i P + \nabla_i \Phi_{\rm N}\right) dV \\
&\quad - \frac{1}{4\pi G}\int_V (\nabla^2\Phi_{\rm N} - 4\pi G\rho)\,\delta\Phi_{\rm N}\, dV - \int_V \xi^i \frac{\partial \rho v_i}{\partial t} dV.
\end{aligned} \tag{B10}$$

Next, we derive a variation of the total angular momentum J defined by

$$J = \int_V \rho\, v_i \phi^i\, dV, \tag{B11}$$

where ϕ^i is a generator of rotation with Cartesian components $\phi^i = (-y, x, 0)$. The variation of J is

$$\delta J = \int_V \rho\, \Delta v_i\, \phi^i\, dV + \int_V \rho\, v_i\, \Delta\phi^i\, dV + \int_V v_i\, \phi^i\, \Delta(\rho\, dV). \tag{B12}$$

Using a relation

$$\Delta\phi^i = \delta\phi^i + \pounds_\xi \phi^i = -\pounds_\phi \xi^i, \tag{B13}$$

the second term of Eq.(B12) is rewritten as follows:

$$\begin{aligned}
\int_V \rho\, v_i\, \Delta\phi^i\, dV &= -\int_V \rho\, v_i\, \pounds_\phi \xi^i\, dV = -\int_V \pounds_\phi(\rho\, v_i\, \xi^i)\, dV + \int_V \xi^i\, \pounds_\phi(\rho\, v_i)\, dV \\
&= -\oint_{\partial V} \rho\, v_i\, \xi^i \phi^j\, dS_j + \int_V \xi^i\, \pounds_\phi(\rho\, v_i)\, dV,
\end{aligned} \tag{B14}$$

where we used $\nabla_j \phi^j = 0$. Discarding the surface term in the above expression and substituting in Eq.(B12), we have a variation of the total angular momentum δJ as follows:

$$\delta J = \int_V v_i\,\phi^i\,\Delta(\rho\,dV) + \int_V \rho\,\phi^i\,\Delta v_i\,dV + \int_V \xi^i\,\pounds_\phi(\rho\,v_i)\,dV, \tag{B15}$$

Finally we write down a general expression for the combination of δE and $\Omega\delta J$, where Ω is a constant parameter,

$$\begin{aligned}
\delta E - \Omega\delta J = \\
&\int_V \rho\,T\Delta s\,dV + \int_V \left(\frac{1}{2}\,v^2 + \Phi_{\rm N} + u + \frac{P}{\rho} - v_i\,\Omega\phi^i\right)\Delta(\rho\,dV) + \int_V \rho(v^i - \Omega\,\phi^i)\,\Delta v_i\,dV \\
&+ \int_V \xi^j v_j \left[\frac{\partial\,\rho}{\partial\,t} + \nabla_i(\rho v^i)\right] dV + \int_V \xi^i \rho \left(\frac{\partial\,v_i}{\partial\,t} + v^j\nabla_j v_i + \frac{1}{\rho}\nabla_i P + \nabla_i\Phi_{\rm N}\right) dV - \frac{1}{4\pi G}\int_V (\nabla^2\Phi_{\rm N} - 4\pi G\rho)\,\delta\Phi_{\rm N}\,dV \\
&- \int_V \xi^i \left[\frac{\partial\,\rho v_i}{\partial\,t} + \pounds_{\Omega\phi}(\rho\,v_i)\right] dV,
\end{aligned} \tag{B16}$$

As an application of the above general expression, consider a Newtonian binary star system in circular orbit [41]. In this case, the fluid variables $\mathcal{Q}$ admit a helical symmetry, namely,

$$\left[\frac{\partial}{\partial\,t} + \Omega\pounds_\phi\right]\mathcal{Q} = 0, \tag{B17}$$

that is, the last integral in Eq.(B16) vanishes. When a Mass conservation equation, the Euler equation and the Poisson equation for the Newtonian gravity are satisfied, namely,

$$\frac{\partial\,\rho}{\partial\,t} + \nabla_i(\rho v^i) = 0, \qquad \frac{\partial\,v_i}{\partial\,t} + v^j\nabla_j v_i = -\frac{1}{\rho}\nabla_i P - \nabla_i\Phi_{\rm N}, \quad \text{and} \quad \nabla^2\Phi_{\rm N} = 4\pi G\rho, \tag{B18}$$

Eq.(B16) takes a simpler form,

$$\delta E = \Omega\delta J + \int_V \rho\,T\Delta s\,dV + \int_V \left(\frac{1}{2}\,v^2 + \Phi_{\rm N} + u + \frac{P}{\rho} - v_i\,\Omega\phi^i\right)\Delta(\rho\,dV) + \int_V \rho(v^i - \Omega\,\phi^i)\,\Delta v_i\,dV. \tag{B19}$$

If we further assume that the perturbed flow is isentropic and mass conserving,

$$\Delta s = 0 \quad \text{and} \quad \Delta(\rho\,dV) = 0 \tag{B20}$$

and that the vorticity of each fluid elements is conserved,

$$\Delta\,\omega_{ij} = \Delta(\nabla_j v_i - \nabla_i v_j) = \nabla_j\Delta v_i - \nabla_i\Delta v_j = 0, \tag{B21}$$

then Eq.(B19) reduces to

$$\delta E = \Omega\,\delta J. \tag{B22}$$

Here we have used Eq.(B21) to introduce a function Ψ for which

$$\nabla_i\Psi = \Delta v_i; \tag{B23}$$

this form of Δv_i, together with helical symmetry imply that the last term in Eq.(B19) vanishes:

$$\int_V \rho(v^i - \Omega\,\phi^i)\,\Delta v_i\,dV = \int_V \rho(v^i - \Omega\,\phi^i)\,\nabla_i\Psi\,dV = \oint_{\partial V} \rho(v^i - \Omega\,\phi^i)\,\Psi\,dS_i + \int_V \left(\frac{\partial\,\rho}{\partial\,t} + \pounds_{\Omega\,\phi}\rho\right)\Psi\,dV = 0 \tag{B24}$$

where we used mass conservation equation and $\nabla_i\phi^i = 0$.

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